THE ELEMENTS

OF

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VOL. I.

ELEMENTARY STATICS AND DYNAMICS.

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THE

ELEMENTS OF NATURAL PHILOSOPHY.

PART I.

INTRODUCTION TO THE MECHANICAL SCIENCES.

CHAPTER I.

OF MATTER AND ITS PROPERTIES.

(1.) THE object of Mechanics being to investigate the effects produced by forces upon material substances, it is necessary, in commencing a treatise on the Mechanical Sciences, to say a few words respecting the properties of matter, and the various forms in which it exists in nature. We shall dwell but briefly on this subject, and only so far as it is necessary to the proper understanding of Mechanics. We shall confine our attention to those general properties which belong, more or less, to all kinds of matter, and upon which the various effects produced by forces, under different circumstances, depend. The special properties which distinguish the different kinds of matter from each other belong to the province of Chemistry, and we shall not make any allusion to them here.

(2.) *Of the Inertness of Matter.*—The terms *Matter* and *Spirit* are to a certain degree opposed to each other; the former being applied to the various substances, whether solid, fluid, or aëriform, of which our own bodies and the world around us are composed; while the latter is applied to that mysterious and unseen substance of whose existence within himself each man is conscious. The distinguishing property of Matter is, that it is perfectly passive, without power of self-locomotion, and incapable of change except by the force of some external cause. On the contrary, Spirit is endowed with that wonderful active principle called the *Will,* which is capable of producing, by an internal and independent effort, various motions and changes in the material body over which it has power. We know by observation and experience, and by reasoning upon the phenomena of the material world, that Matter has no self-active energy; and we are convinced by what we feel and know of ourselves, that Spirit is self-active, by which we mean that it has the power of choosing to act or not to act, independently of any external influence.

It is true that material bodies exert various powers upon each other; thus the Sun has the power of drawing or attracting the Planets towards him, and the Planets exercise the same kind of power upon the Sun, and upon each other. But we know of no material body that can act upon itself so as to put itself in motion if it be at rest, or to stop its motion if it be in motion. We, therefore, assume this want of self-active power to be the distinguishing property of matter, and *this is what is* meant by saying that matter is

inert, or that it possesses the property of *Inertia*, the word being derived from the Latin, and signifying inactivity or sluggishness. We must say more respecting the full meaning of this important word when we come to consider the subject of Motion.

(3.) *Extension and Impenetrability.*—A material substance always occupies a certain portion of space, and this is called the property of extension. Furthermore, one body cannot penetrate or occupy the same space as another without, to a certain extent, displacing it: this is called the property of *impenetrability.*

(4.) *Divisibility.*—Every material body may be divided into two or more, or any number of parts, and, as far as our experience goes, there is no limit to this divisibility of matter. We may, without difficulty, divide a grain of gold into at least a million of parts visible to the naked eye; and one of these parts might doubtless be divided again into as many subdivisions. How far the process of subdivision may be carried in the case of any substance we cannot assert, we can only say that there is no sensible limit to the degree of minuteness to which the division of matter may be effected.

There is good reason, however, to suppose that all material substances are composed of what are called *Atmos,* by which word is meant "that which cannot be *cut*, or *divided.*" These atoms are often called *Particles,* from the Latin word signifying a *very little part*; they are also called *Molecules* and *Corpuscules,* both which words mean very minute bodies. They are supposed to be inconceivably small; indeed some philosophers have imagined, and with reason, that they are actually devoid of

extension, being nothing but mere points endowed with powers of attraction and repulsion, in virtue of which they collect together, without coming into actual contact, and so form masses of extended matter. It is almost proved, we may say, that there is a great deal of truth in this view of the constitution of matter, viz. that bodies are really composed of particles which may be regarded as indivisible, or rather, which never are divided in any process of nature. Whether they are mere points, or absolutely incapable of division by any means, it is not necessary, nor possible, to determine.

(5.) In Mechanics the words Particle, Corpuscule, Molecule, are continually employed, without, however, any assumption of their indivisibility; in fact, we do not call them atoms. All that we mean by these terms is, that we may regard bodies as composed of an immense number of minute parts, to which parts we give the above names. A very good term is used in the same sense by some writers, namely, *Material Point*, by which is meant, a portion of a material body so small that it may be considered as a point without sensible error. Giving this meaning to the term, we may suppose every body to be composed of material points adhering together.

(6.) *Porosity.*—A substance is said to be *porous*, or to possess the property of *Porosity*, when it is full of holes or pores, like a sponge. Every kind of matter that has been experimented upon appears to be *compressible*, that is, capable of being forced into a smaller space than it naturally occupies. This fact alone is sufficient to make the porosity of matter a very reasonable supposition, for compressibility is a natural consequence of porosity.

But the mixing together of substances, as, for instance, fluids with fluids, appears to leave no doubt of the porosity of matter. How, for instance, could wine and water be mixed together so intimately as to become, as it were, one fluid, if they did not mutually penetrate into each other in virtue of their porosity? The chemical union of substances, as, for instance, the composition of water by the union of the two gases, oxygen and hydrogen, is perhaps the best proof that can be given of this property of matter.

Philosophers account for the porosity of matter very simply, by saying, that the atoms or particles are not in contact with each other, but are kept at certain distances from each other by forces of attraction and repulsion, as we have observed before. Of course, if matter is composed of particles not in contact with each other, it is necessarily porous.

(7.) *Cohesion and Repulsion.*—That the particles of solid bodies stick together, or *cohere*, is familiar to every one, inasmuch as it requires a certain amount of force to break or tear asunder such bodies, which would not be the case if the particles had not a power of cohesion. This power is often called the *Attraction* or *Force of Cohesion.* In some bodies it acts powerfully, in others feebly; and this is, in a great measure, the cause of those different qualities which we call hardness, toughness, softness, fluidity, &c. In the case of fluids, the power of cohesion appears to be almost insensible, compared with what it is in solids.

But the particles of matter are also endowed with a power of *repulsion,* which makes itself manifest when we attempt to compress bodies. We must use force to compress a body into a

smaller space than that it naturally occupies; and in many cases, the body in a great measure recovers its original dimensions when the compressing force is removed. Every known substance has this power of resisting compression, in a greater or less degree; from which fact we conclude that the particles of matter repel each other when they are brought closer together than in their ordinary condition.

These two powers of cohesion and repulsion are often supposed to be merely modifications of the same force, which has been called *molecular force*. This force is supposed to be exerted between molecule and molecule, and to be an attractive or repulsive force according to the distance between the molecules. When the molecules are at a certain distance from each other, the molecular force does not act; when at a greater distance it becomes attractive, and at a smaller distance repulsive. A supposition of this kind fully accounts for the facts above stated respecting the resistance which bodies offer either to a tearing or breaking force, or to a compressing force. The supposition, however, of a cohesive and repulsive power residing in each particle comes really to the same thing, and is a simpler way of explaining the phenomena of cohesion and repulsion to persons not familiar with mathematical formulæ.

We may assume then, that, when the particles of a body are at their natural and unconstrained distance from each other, the cohesive and repulsive powers are equal, and destroy each other; but when the distance is increased the attractive power *prevails*, and when it is diminished the repulsive *power prevails*. When the distance is consider-

ably increased both powers appear to become extinct, as we know from the fact, that, when once a fracture is made in a body, there is no difficulty in removing the two parts to any distance from each other.

(8.) The distance between the particles at which the two opposing powers of cohesion and repulsion destroy each other, depends very much upon the amount of heat in the body; the greater that amount is, the greater the distance becomes. In fact, heat appears to increase the repulsive power, or, what would be the same thing, to diminish the attractive power. The consequence of this is, that heat expands, and cold contracts bodies. When the heat of a body is increased to a certain degree, the cohesive power is so much diminished that the repulsive power causes the body to expand into a vapour, as we know from common experience. It is probable that heat is itself the repulsive power which keeps the particles from coming together.

(9.) *The three Forms of Matter: Solid, Liquid, and Aeriform.*—Material substances occur in three distinct and dissimilar forms or conditions. *First* of all there are *solid substances*, or those which possess in a considerable degree the property of hardness. In such substances the powers of cohesion and repulsion both act with considerable energy, and make it difficult either to increase or diminish the distance between the particles; so that the body offers considerable resistance if we attempt either to tear it asunder or to compress it. The various degrees and qualities of hardness which we find in solid bodies depend upon the relation between the cohesive and repulsive powers. *Secondly, there are liquid substances.* These appear

to have little or no cohesive power, but quite as much repulsive power as solid bodies, or rather more. If the distance between the particles be ever so little diminished, the repulsive power acts strongly; but, if increased, little or no cohesive power is brought into play. *Lastly,* there are *aeriform substances,* gases and vapours of various kinds. In these the cohesive power appears to be utterly extinct, and the repulsive power comparatively weak, though strong enough to make itself sensible by the tendency which the substance has to expand and diffuse itself in space, if not restrained and kept in by some force or resisting obstacle or vessel. We say that the repulsive power is comparatively weak, for a moderate amount of force is able to compress a gas into half or quarter the space it occupies; whereas no force that we can command, however great, is capable of doing this, or anything approaching to it, in the case of a liquid or solid substance, if we except very soft and porous solids.

By the action of heat we may, as we have stated, greatly modify the intensities of the cohesive and repulsive powers, and so change solids into liquids, and liquids into vapours: and the reverse action may be produced by cold, as has been remarkably exemplified in the case of several gases, which resisted all efforts to compress them into the liquid form, until a considerable degree of cold was produced previously.

(10.) *Rigidity, Flexibility, Elasticity, &c.*—When the cohesive and repulsive powers are so energetic that no ordinary amount of force can increase or diminish the distance between the particles of a body, it is called a rigid body. A rigid body,

therefore, is one which cannot be broken, or bent, or compressed; in short, one whose shape and dimensions cannot be altered by the action of ordinary forces. There is no such thing as a perfectly rigid body in nature, though many bodies may be regarded as rigid when not exposed to the action of unusually great forces. In practice it is always necessary to bear in mind that no substance is so rigid that it cannot be bent or broken; and it is an important part of Mechanical Science to determine how far, and within what limits, different kinds of materials can resist the action of forces tending to bend or break them.

(11.) A remarkable property of many substances is *flexibility*, or the capability of being bent. A string or rope is a familiar instance of bodies possessing this property. The least force will bend a string, and make it assume any curved form that we please. A piece of wire possesses this property also, but in a much less degree, for it requires some force to bend it. No body is perfectly flexible, it always requires some little force to bend even the most flexible body. There is no substance that does not possess a certain degree of flexibility, inasmuch as there is no substance that is perfectly rigid, for if the rigidity of a body is imperfect, it must be flexible to a certain extent.

There are some flexible bodies, (as for instance, a common rope,) which cannot be stretched or broken except they are pulled by a considerable force. Such bodies are said to be *inextensible*, that is, not capable of being extended or stretched. When we talk of a string or rope being inextensible, we do not mean that it is perfectly so, for there *is no substance that* cannot be stretched if

a sufficiently powerful force be employed to pull it; we only mean that it is not capable of being stretched in any sensible degree by the action of moderate forces.

Again, there are other flexible bodies which offer but little resistance to a pulling force, and may be easily stretched, and that to a considerable extent, without breaking. A string of Indian rubber is an instance of this. Bodies of this kind are said to be *extensible.*

(12.) When a body is compressed, or stretched, or bent by the action of a force, it is said to be *elastic,* if it recovers its natural shape and size on the removal of the force. If a string of Indian rubber be stretched, the moment it is let go it springs back and assumes its original length. If a straight steel spring be bent into a curve, it becomes perfectly straight again, when released from the bending force. If air be compressed in a vessel into a smaller space than that it naturally occupies, it will expand and resume its original bulk when the compressing force is removed. All these bodies are said to possess the property of *elasticity,* which is a power of quickly recovering from the effects of a compressing, stretching, or bending force.

When a body is compressed by a force, and then allowed to expand again by removing the force, it exerts a certain amount of force in the act of expanding, which may be felt if we try to prevent that expansion. This force of expansion is sometimes nearly equal in amount to the force which compressed the body, and sometimes considerable less than it, but never either quite equal *to, or greater than* it. When the force of expan-

sion is nearly equal to the compressing force, the body is said to be very elastic, or to possess the property of elasticity in a high degree; and, in general, the nearer the force of expansion approaches in amount to the compressing force, the more elastic the body is said to be.

(13.) *Heaviness and Lightness.*—One of the most remarkable properties of material substances, and familiar to every one, is a tendency to fall down, commonly called *weight* or *heaviness*, which appears to be greater in some kinds of substance, and less in others. This tendency to fall down is made manifest to our senses by the actual falling of bodies when unsupported, and by the muscular effort we must make to prevent bodies from falling down. It is chiefly, however, from the muscular force required to hold up bodies, that we get the idea of weight or heaviness; and, indeed, we can form a good idea of the relative weights of different bodies, by supporting them in the hand, and feeling what amount of muscular exertion is required to do so.

But there are some substances which appear to possess the opposite property of *lightness*, or a tendency to rise. Thus, the bag of a balloon being filled with the gas called hydrogen, shows a very considerable tendency to rise, and is capable of drawing up a considerable weight. Other substances, such as feathers, smoke, or the like, appear to be neither heavy nor light—that is, they seem to have no tendency either to rise or fall. But what is most remarkable is, that the heaviness of a body appears to be capable of alteration by immersing it in different fluids. Thus, a piece of wood, which exhibits a tendency to fall in air, if

put under water, immediately rises, and a piece of iron, which, when put in water, immediately sinks, will exhibit a considerable tendency to rise, if immersed in quicksilver.

The phenomena of heaviness and lightness are now well understood, and there is no difficulty in explaining the facts just stated in a very simple manner by the theory of gravitation, as we shall presently show; but not long ago this subject was one of the greatest difficulty to philosophers, and gave rise to continual disputation.

(14.) We have now said enough, by way of introduction to the Mechanical Sciences, respecting the general properties of matter. To most of the points here alluded to, we shall have to recur hereafter at some length. We shall conclude this chapter by mentioning facts and experiments which confirm and illustrate some of the above statements.

EXPERIMENTAL ILLUSTRATIONS.

(15.) *Divisibility of Matter.*—If we take a grain of one of the common blue dyes, (a compound of copper,) and put it in a gallon of water, the whole of the water will become sensibly coloured blue. Now, it would be easy to show that there are more than 1,000,000 small drops of water in a gallon; if, therefore, we take one small drop of the coloured water, it will contain only the millionth part of a grain of the dye. In this way we may divide so small a portion of matter as one grain of the dye into a million of parts. But, furthermore, the drop of coloured water, seen in a powerful *microscope, would* appear of considerable size, and

capable of being divided into some 1,000 visible equal parts. This proves that the grain of dye might be divided into a thousand millions of equal parts.

The animalculæ seen in a powerful microscope afford a wonderful proof of the extreme divisibility of matter. It has been calculated that more than a million of animalcules of a certain kind, heaped together, would form so minute a portion of matter, as to be scarcely visible to the naked eye. Now, each of these creatures is an organized being, having limbs of various kinds, and vessels or tubes for the circulation of fluids. How inconceivably minute the particles of the fluids which circulate through these tubes, and the particles of which the creature is composed must be.

(16.) *Porosity of Matter.*—Quicksilver may be easily forced in minute drops through leather, wood, and other substances of the same nature; which proves the porosity of these bodies. The well-known experiment tried at Florence in the seventeenth century shows that gold, apparently one of the most condensed and closest substances, is porous, and that water may be forced through its pores. The Florence academicians filled a hollow ball of gold with water, and then put it in a press, where it was exposed to the action of a considerable compressing force, in order to try whether water was a compressible substance or not; for they supposed that gold was too dense a substance to allow the water to pass through it, and that they might succeed in this way in compressing the water into a smaller space than that it naturally occupied, by squeezing the gold ball in *the inside of which it* was contained. The

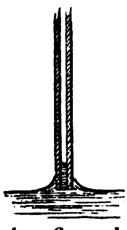
result of their experiment was quite contrary to their expectations, for the water was actually forced through the pores of the gold, and appeared like dew on the outside of the ball. The fact that water may be forced through the pores of gold has been frequently proved by experiment.

(17.) *Forces of Cohesion and Repulsion.*—If two pieces of plate glass, soon after they have been polished, be placed with their surfaces in contact, they will soon stick together, and, if left for any time in such a state, will become as closely united to each other as if they had been originally only one piece; so that if an attempt be made to force them asunder, it will be found that they will not split or give way along the surface of union, as one might fancy, but in some other direction. This has often occurred in plate-glass factories, and it affords a very striking illustration of the powerful force of cohesion which the particles of glass are capable of exercising on each other.

The forces of cohesion and repulsion exist not only between particles of the same kind, but also between particles of different kinds, as the following facts will show:—

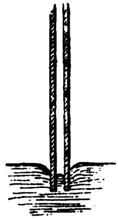
If a small open glass tube be dipped in water, as in fig. 1, the water will rise in the tube, and outside it, as is represented in the figure. This proves that the particles of glass which compose the tube draw up the water towards them, which will be made very manifest if we lift the tube slowly and gently out of the water. In other words, this experiment shows that there is a cohesive force between the particles of glass and water.

Fig. 1.



If the same sort of tube be dipped in quicksilver instead of water, the contrary effect will be produced, as is shown in fig. 2. The quicksilver will sink, instead of rising, both inside and outside the tube; which shows that there is a repulsive power between the particles of glass and quicksilver, which makes them recede from each other, and in this way the quicksilver is forced downwards.

Fig. 2.



(18.) *Why certain substances are wet by certain fluids, and others not.*—Water wets glass, (if it be not greasy,) because there is a cohesive force between the particles of water and glass, in virtue of which the water adheres to, or wets the glass. The same may be said of wood and water. Quicksilver, on the contrary, does not wet glass, because there is a repulsive force between the particles of glass and quicksilver, which prevents the quicksilver from adhering to the glass; and the same may be said of water and any greasy substance.

(19.) Put two balls, one of wood, and the other of any greasy substance, or of wax, in a vessel of water, (fig. 3,) and the contrast of the cohesive and repulsive forces will be made very manifest. The water will rise and wet the wooden ball, as is shown in the figure, and will so make an elevation on the surface of the water. On the contrary, the water will not wet, but will be repelled by the wax ball, and the consequence will be, that a depression will be formed in the surface of the water.

Fig. 3.

This *explains the reason* why balls put in water

in this way, sometimes attract, and sometimes repel each other; and why little substances floating on the surface of a fluid in a vessel, sometimes run to the sides of the vessel, and sometimes do not. When two balls of wood are placed sufficiently near each other in water, the water rises between the two balls considerably more than it would do, if there was only one ball, as is represented in fig. 4, and the effect of this is, that the balls are drawn together by the cohesive power of the water which rises up between them. Again, if the two balls be of wax, the water sinks between the two balls more than it would do if there were only one ball, as is represented in fig. 5, and the consequence is, that the water at *a* and *b*, by the repulsive power it exerts on the balls, pushes them together.

Fig. 4.

Fig. 5.

If one ball be of wood, and the other of wax, they are pushed away from each other by the action of the water. The reason of this may be given in the same manner as in the former cases; and the same may be said of the fact, that substances floating near the sides of the vessel which contains it, sometimes run towards the sides, and sometimes do not.

(20.) *Solids, Liquids, and Gases.*—It is scarcely necessary to give instances of the three forms which material substances assume, but it is worth while stating a few facts relative to the change of form which is brought about by the action of *heat* or cold, and by external pressure.

The degree of heat, or the *temperature*, as it is

called, to which a mass of water must be raised to make it boil, is, as most people know, 212° of Fahrenheit's thermometer, under ordinary circumstances. Now, the water begins to boil, or assume a gaseous form, as soon as the cohesive power of its particles is so far diminished by the action of heat, (as we have stated above,) that the repulsive power prevails sufficiently to make the particles tend to separate, and fly away from each other with rapidity. But this tendency is restrained by the pressure of the atmosphere on the surface of the water; for the atmosphere, as we shall presently explain more particularly, exercises a considerable pressure on every body exposed to it, amounting, in round numbers, to fifteen pounds on every square inch. It is easy to understand that so great a pressure as this must keep in the particles of the water, and overcome their tendency to fly away from each other, unless there be sufficient heat communicated to increase the repulsive power so much as to make it overcome the atmospheric pressure. As we have stated, it requires a temperature of 212° Fahrenheit to produce this effect; and we may easily see why the water, which, comparatively speaking, is not very much affected at a lower temperature, flies rapidly into vapour as soon as the temperature comes up to 212°. Under this temperature the repulsive tendency communicated to the particles of the water by the heat is completely kept in check by the superior atmospheric pressure, and no boiling takes place; but as soon as the repulsive tendency is increased so much as to exceed the atmospheric pressure, there is nothing to restrain the *separation of the particles* of the water, and

the consequence is, that the water turns rapidly into steam, or, in familiar language, boils.

(21.) This view of the phenomenon of boiling by the action of heat is strongly borne out by the fact, that, if the atmospheric pressure be diminished, the water boils at a lower temperature than 212°; which may be shown either by boiling water on the top of a mountain or under the receiver of an air-pump. At the top of a mountain of some elevation, the atmospheric pressure is considerably less than at the level of the sea: and, accordingly, it is found that water boils at a lower temperature on the top of the mountain than at the level of the sea. Indeed, it has been proposed to measure the height of mountains by observing at what temperature water boils upon them; and an instrument for this purpose is often used when very great accuracy is not required.

(22.) If we boil water under the large glass vessel in the air-pump called the receiver, from which the air is pumped away, and in which therefore there is little or no atmospherical pressure, it is found that the temperature at which boiling commences is so low as 72° of Fahrenheit's thermometer, while in the open air it requires a temperature of 212°. This is an important experiment, inasmuch as it shows that when the atmospherical pressure acts on the surface of the water, it requires as much as 140° increase of temperature to overcome its restraining power, and make the water boil.

(23.) Pressure, therefore, counteracts the effect of heat in turning liquids into a gaseous form; and of course, cold, on the contrary, must be assisted by pressure in making gases assume a *liquid form.* Cold alone can produce the effect

of turning vapours into liquids, as we know in the case of steam and other vapours; but there are many gases which no known degree of cold will condense into liquids without the assistance of pressure; nor will any known amount of pressure, without the assistance of cold. By combining cold and pressure Dr. Faraday has succeeded in turning into liquids several gases that were supposed to be incapable of such a transformation.

(24.) A simple experiment, to show the effect of diminished pressure in making water boil at a lower temperature, is thus made: partially fill a flask or bottle with hot water, and make it boil with a spirit-lamp, or otherwise; cork the flask tightly, and allow the water to cease boiling for some minutes, and then plunge the flask in cold water. The effect will be curious; for the water in the flask will immediately begin to boil again, though the temperature has really been diminished by putting the flask into cold water. The explanation of this is easy; for the vapour which fills the portion of the flask not occupied by water exercises a pressure on the surface of the water equal to that the atmosphere would exert if the flask were not corked; but the cold of the water into which the flask is plunged immediately condenses this vapour into water, and so relieves the water in the flask from pressure; and thus, the pressure being removed, and the temperature being considerably over 72°, the water immediately boils.

(25.) *Tenacity of Materials.*—When a string or wire is exposed to the action of a stretching force of sufficient power, it is torn asunder. The

greater the force which is required in order to do this, in proportion to the thickness of the string or wire, the more *tenacious* the substance is said to be. The term *tenacity*, meaning the degree of resistance which the substance is capable of opposing to a force applied to tear it asunder, is estimated by the force required to tear asunder a wire of the substance of a certain thickness. Iron is a substance possessing a wonderful degree of tenacity. A bar of cast-iron, whose transverse section, or thickness, is a square inch, will support more than eight tons without breaking; a similar bar of wrought-iron will sustain over twenty-five tons without breaking; a cable of iron wire, of a square inch thickness, will support as much as sixty tons without breaking.

(26.) *Heaviness and Lightness.*—There is an old experiment which may be mentioned in relation to this property of matter. In the open air, a guinea appears to be composed of a much heavier substance than a feather, as we generally conclude from the fact, that if we let both bodies go at the same time, the guinea will fall very quickly to the ground, but the feather very slowly, or not at all. If, however, we remove the air, which may be done by allowing the bodies to fall inside a tall receiver of an air-pump from which the air has been pumped, the feather will fall quite as quickly as the guinea, and therefore the substance of the feather will appear to be as heavy as that of the guinea. We shall explain the reason of the lightness, or buoyancy, of some bodies in air, water, and other fluids, hereafter.

CHAPTER II.

OF FORCE, AND THE VARIOUS KINDS OF FORCE IN NATURE.

(27.) *Of the term Force.*—It is not necessary to define the term *force*, which we have frequently introduced in the preceding chapter, inasmuch as its meaning is familiar to every one. Indeed, the idea expressed by this term is acquired by the mind in the same manner as the ideas of *tone*, *taste*, *colour*, and others of the same nature, which are called *simple ideas*. Ideas of this kind can be conveyed to the mind by *exemplification*,—that is, by giving examples or instances,—but not by definitions properly so called. In explanation, then, of the meaning of the term *force*, it is enough to say, that when we pull a body by a string, we exert a *force* upon it; when we push against an obstacle, we exert a *force* upon it; when we hold up a heavy body, we exert a *force* upon it; and, in like manner, when any other agent, animate or inanimate, exerts a similar power, by pulling, or pushing, or supporting, or the like, we say that that agent exerts a *force*. It is usual, however, to define force to be "*that which produces* or tends to

produce motion;" but it may be questioned whether these words would convey anything like the idea of force to a mind previously devoid of that idea.

(28.) The idea of force appears to be suggested to the mind by the muscular effort we have to make when we endeavour to move a body, or prevent its motion: by this effort we become sensible that we exert a force upon the body. In the same way we become sensible of the forces which other agents exert, by the effort it requires to overcome these forces, or prevent them from taking effect. Thus, it requires a certain amount of muscular exertion to hold up a heavy body, and keep it from falling: this exertion makes us sensible that there is some force pulling the body downwards. In like manner, the effort required to break a tough body across makes us feel that there is a force of cohesion keeping the parts of the body together.

(29.) *Of the different kinds of Force in Nature.*—Before commencing the study of mechanics, properly so called, which consists of various propositions and investigations relative to the action of forces, singly and in combination, it is desirable that the student should have some knowledge of the different kinds of force which act in the material world, and of the means and contrivances by which they are estimated and measured. Such preliminary knowledge is not only useful for its own sake, but it also has the advantage of making familiar to the student the idea of force in general, the nature of the effects it produces under various circumstances, and several of the terms employed in relation thereto; so that, when he commences

the formal and abstract reasonings which occur at the commencement of and throughout mechanics, he finds little or no difficulty in understanding them. The reason why mechanics sometimes appears dry and difficult is, because the student has never thought upon the subject in a familiar and simple way, before commencing the formal and abstract study of it. We shall, therefore, devote the present chapter to an enumeration of some of the various kinds of force in nature, and an explanation of methods employed to estimate their energy.

OF THE ATTRACTION OR FORCE OF GRAVITY.

(30.) We shall commence with the Force of Gravity, which produces effects so familiar to us and of such constant occurrence, and whose sphere of action extends throughout the visible universe. The distances of the bodies composing the solar system from the sun and from each other are regulated and preserved within due limits by the agency of this wonderful force, and there is good reason for asserting the same of the whole host of stars which surround us. The particles of each body are held together in compact globular masses by the force of gravity; and, in our own world, the multitude of objects, animate and inanimate, which occupy the surface of the earth, are by the same force kept in their proper places and positions. Indeed, it would be an endless work to enumerate the uses and consequences of the action of this force.

(31.) *Of the Attraction of Gravity generally.*— The *Theory of Universal Gravitation*, as it is

called, established by Newton in such a manner that there can be no doubt of its exact truth, is, that every particle of matter is endowed with a power of *attracting*, or drawing towards it, every other particle of matter, whether near or at a distance; but the greater the distance, the less is the power of attraction, according to a certain law which we shall presently state. Thus, if A and B, fig. 6, represent any two particles of matter, A attracts B, and B attracts A; so that the two will run together, except there be some obstacle or repulsive force to prevent it. Furthermore, the nearer the two particles are placed, the more energetic will this force of attraction be, and the more violently will the particles run together, if allowed to do so; and, *vice versâ*, the greater the distance, the less the attraction. Furthermore, the amount of force which A exerts upon B is precisely the same as that which B exerts upon A; or, in other words, it will require just as much resisting force to prevent A moving towards B as it does to prevent B moving towards A. And this is true whether A and B be equal or unequal in size.

Fig. 6.

With regard to the effect which the distance between the two particles has upon the amount of attraction they exercise upon each other, we shall only observe now, that if the distance is diminished, and made one half of its original length, the force of attraction is increased to so much that it becomes four times greater than it was before; if the distance is made one-third, the attraction becomes nine times greater; and so on, according to the law expressed in the following table, sup-

posing F to denote the force of attraction when the distance between A and B is 1.

Distance between A and B.	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	&c.
Force of attraction.	F	$4\,F$.	$9\,F$.	$16\,F$.	$25\,F$.	&c.

The numbers in the second line here, as is evident on inspection, are the *squares* of 2, 3, 4, 5, &c. Hence, the attraction of gravity is said to increase as the distance diminishes, *in inverse proportion to the square of the distance.*

(32.) This is the statement of the theory or law of universal gravitation; it applies to every particle of matter composing the earth, and every thing upon it, not excepting the air that surrounds it; it likewise applies to every particle composing the solar system, and we have good reason to assert the same of the whole visible universe. We shall hereafter state the proofs which Newton has given of the existence and universality of the law of gravitation, as far as the solar system is concerned. Since his time, various motions observed among the stars, and the establishment of the fact that the sun, followed by the planets, is moving rapidly in space, afford strong grounds for the assertion that this law extends to the fixed stars. We shall hereafter show, that what we now state furnishes a most wonderful and stupendous evidence of unity and design in the visible universe.

(33.) *Weight or Heaviness caused by the Attraction of Gravity.*—We have already spoken of the tendency to fall which most bodies manifest, and

which is familiar to every one under the name of *weight* or *heaviness.* The reason why a body tends to fall, is because the attraction of gravitation residing in the particles of matter which compose the earth draws the body towards the earth, and produces in it a tendency to fall. Every particle composing the earth attracts the body, and the effect resulting from all these attractions is, that the body is drawn towards the centre of the earth, or very nearly so. Properly speaking, the body is attracted *perpendicularly towards* what is called *the mean* or *level surface* of the earth; that is, what would be the surface, if there were no mountains or valleys; or, to speak more accurately, if the sea covered the whole earth, and were undisturbed by winds or tides. Now, this level or mean surface of the earth is nearly, but not exactly, a sphere; and, therefore, the perpendicular to it does not in all places point directly towards the centre of the earth, though the deviation from the centre is very small. We shall not at present be so unnecessarily exact as to take this deviation into account; and we shall therefore say that the body is drawn, by the aggregate attraction of all the particles of the earth, towards the earth's centre.

(34.) The ancients, not having had any definite idea of the earth's rotundity, had very confined notions of the meaning of the words *up* and *down;* and the idea of *antipodes,* where men were obliged to walk with their feet upwards and heads downwards, was very difficult of comprehension. We now understand the words up and down in an enlarged sense: *down* means, towards the earth's *centre, and up,* of course, the contrary direction;

or, speaking more accurately, *down* means the direction in which bodies fall towards the earth. Hence, since the earth is round, and bodies always fall towards the centre of the earth, it follows that *down* here is about the same direction as *up* in New Zealand. One of the great objections brought against the supposition of the earth's being a round ball was, that it made it difficult, if not impossible, to distinguish between *up* and *down*; for what was *up* to one man, would be *down* to another on the opposite side of the earth, if it were a globe. The force of this difficulty cannot be estimated in our days, when the rotundity of the earth and the idea of antipodes are familiar to all; but in former times it was very different, and the difficulty was perpetually urged by the most enlightened men. We may notice here the importance of a correct definition, for, by defining the word *down* as the direction in which bodies fall towards the earth, all the difficulty we allude to vanishes.

(35.) Returning to the subject of *heaviness*, we are now prepared to state the cause of it very simply: it is the effect of the attraction we have just spoken of towards the centre of the earth. Every body is drawn downwards (*i.e.* towards the earth's centre, according to our definition of the word) by the attraction of the earth; and the tendency to fall so produced is called *heaviness*.

(36.) By the term *weight* we mean somewhat the same thing; but, to speak more correctly, and attach definite ideas to our terms, we must define the weight of a body to be *the amount of force which acts upon it in consequence of the earth's attraction of gravity*. *Weight*, then, is the force resulting from

the earth's attraction of gravity. It is, therefore, a force which always acts downwards towards the centre of the earth, and draws every body towards that point.

(37.) *Vertical Line; Horizontal Plane.*—We may here conveniently define the terms *vertical* and *horizontal.* By a *vertical line* is meant a line drawn in the direction in which bodies fall, or, what is the same thing, in which weight acts. This, as we have stated, is a line drawn perpendicular to the earth's level surface; or, what is very nearly the same thing, a line drawn towards the earth's centre.

We may easily find the vertical direction experimentally by letting a heavy body fall, or, what is much better, by hanging a heavy body by a string. The string, of course, will hang downwards in the direction in which the force of gravity pulls the heavy body, that is, in the vertical direction, and thus it exhibits to the eye the vertical line. A string thus used is called a *plumb-line,* from *plumbum,* lead, because the heavy body suspended by the string is generally a piece of lead.

The *horizontal plane* is the plane to which the vertical line is perpendicular. Any line in the horizontal plane is said to be a horizontal line; in other words, all lines at right angles to the vertical line are called horizontal lines.

OF THE CENTRE OF GRAVITY.

(38.) We have stated that the weight of a body is the amount of the force of gravity exerted on that body; but we have not said at what part, or on what point of the body this force acts. In reality, *the weight of* a body arises from the combined

action of several forces, namely, the forces of attraction acting on each particle of the body towards the centre of the earth; for the law of gravitation is, that *each particle* of a body is drawn towards the centre of the earth by the aggregate attraction of all the particles of the earth. Hence, the weight of a body is the total force resulting from the united energy of all the attractions acting on the different particles; or, to speak more simply, the weight of a body is the force resulting from the combined action of the several weights of the particles of the body.

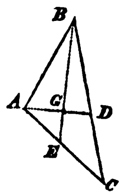
Now, it may be shown, and we shall show it in a future part of this treatise, that this total force arising from the weights of the different particles is equal to the sum of all those weights, (as, indeed, we may assert without proof,) and that it acts at a certain *invariable* point of the body. By an invariable point we mean a point that does not change its position in the body, when the body is turned round, or inverted, or otherwise moved.

This point is called the *centre of gravity* of the body. It may be found by experiment, or theoretically, and the knowledge of its precise position in every body is a thing of great importance. When a body is suspended by its centre of gravity, it will rest in any position it is placed in; for if the centre of gravity, or the point where all the weight acts, be supported, there can be no tendency in the body to move.

(39.) The centre of gravity of any body of perfectly regular form (such as a sphere, a cube, a parallelopiped, or a flat body in the form of a circle, square, parallelogram, or the like) is easily determined, for it is always the central point of the

body, as, for instance, the centre of the circle or sphere, the middle point of the square or parallelogram. In bodies of regular form, there is always a middle point or centre; and it has this distinguishing property, that *every* line drawn through it, to meet the surface of the body both ways, is *bisected* by it. Such a point is always the centre of gravity of the body.

Fig. 7.



The centre of gravity of a flat body in the shape of a triangle *A B C*, fig. 7, is found by drawing lines from any two of the angular points—*A* and *B* suppose—to the middle points *D* and *E* of the opposite sides. The point *G*, where the two lines intersect, is the centre of gravity of the triangle. If we suspend the triangle by a string, or otherwise, at the point *G*, it will remain at rest, without any tendency to turn round, in whatever position we hold the triangle.

(40.) It is important to bear in mind these distinguishing characteristics of the centre of gravity; no other point of the body possesses the same properties. They may be all briefly stated as follows: the centre of gravity does not change its position in the body, when the position of the body is altered in any way whatever. The effect of the force of gravity on the body is to pull the centre of gravity towards the centre of the earth; but it has no tendency to turn the body round, or make it revolve about its centre of gravity.

(41.) From what we have just stated, it follows that the centre of gravity will always assume the *lowest position* it possibly can; for it is the point

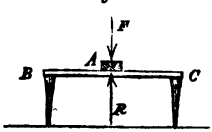
on which the weight acts, and which is drawn downwards by the force of gravity; and therefore, it will always fall, or move to a lower level, if it can, but never rise.

Hence, if we suspend a body by any point, the body will hang with the centre of gravity *vertically below* the point of suspension—that is, both points will be in the same vertical line, the former below the latter. This is obvious from the consideration, that the lowest position the centre of gravity can assume, is when it hangs vertically below the point of suspension.

OF THE FORCE OF RESISTANCE EXERCISED BY SOLID BODIES.

(42.) When one solid body is pressed against another by any force, there is a force of *resistance* brought into play between the two bodies, which prevents them from penetrating into each other. Thus, suppose a body A to be placed on a table $B\ C$, (fig. 8,) and to be pressed against it by any force represented by the arrow F; then the table will resist the action of the force, and prevent the body from being driven downwards. This force of resistance which the table exerts on the body is an upward force, which we have represented by the arrow R. The effect of the force R is simply to destroy the force F, and therefore the two forces must be equal and opposite; for two forces acting on a body with equal energy in opposite directions, can produce no motion one way or the other, which is what we

Fig. 8.



mean when we say, that the force *R* destroys the force *F*. A force unresisted produces motion, but if it be resisted, so that no motion results from its action, it is said to be destroyed. The only way in which a force can be resisted, so as to be destroyed in this sense, is by the action of an opposing force of equal energy; and this is the manner in which the table acts upon the body; it brings into existence an opposing force *R*, which, acting against the force *F*, and with the same energy as *F*, so that the two forces mutually destroy each other, prevents motion from taking place.

(43.) The resistance of a solid body has its limits, which depend upon the toughness and hardness of the material of which the body is composed. We may increase the force *F* in the above example up to a certain amount, and the table will always produce a force *R* capable of destroying *F*; but if *F* be made to exceed that limit, the table will give way, either by breaking, or being crushed, or otherwise, and the body will move down. The table will always furnish a force of resistance *R* just sufficient to destroy the force *F*, whether *F* be small or great, provided it be under a certain limit.

This, then, is the peculiar nature of the resistance which a solid body exerts, when another body is pressed against it by any force not exceeding a certain limit; the resistance will always adapt itself, so to speak, to the pressing force, and be just sufficient to destroy it. There are a great number of problems in mechanics in which this consideration, properly made use of, greatly facilitates the solution.

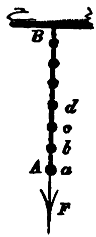
(44.) Reaction.—A force called into action by

another force is called a *reaction;* thus, if a man presses against an obstacle, the obstacle will, as we have just explained, resist that pressure by *returning,* so to speak, an equal and opposite pressure: the *return* pressure thus exerted is called a *reaction.* A reaction is therefore only another name for the force of resistance we have just described.

OF TENSION AND THRUST.

(45.) *Tension,* as the name signifies, has reference to a chain or string drawn tight; it is used to denote the force which is transmitted along a string or chain, when one end is pulled. Let us suppose that *AB* is a chain, and *abcd,* &c. the successive links of it, (fig. 9); and let a force, represented by the arrow *F*, act on this chain at *A*, and pull it, the other end *B* being fastened to some obstacle. Then the link *a*, being pulled by the force *F*, will pull the next link *b* with an equal force, and, in like manner, *b* will pull *c*, *c* will pull *d*, and so on. In this manner the force *F* is transmitted along the chain from link to link, and it is thus brought to bear upon the obstacle *B*. The same might be said of a string or rope of any kind, only, instead of the links, we should speak of the different particles which, united together, form the string or rope.

Fig. 9.



(46.) It is important to notice that the force or tension thus transmitted by the string, always acts along the string, that is, the direction in which the force acts always coincides with the direction of *the string. This is true* also when the string

passes over pulleys, and so has its direction changed. Thus, let *ABCDE* be a string, (fig. 10,) passing over fixed pulleys, *B*, *C*, *D*, which are supposed to allow the string to run over them with perfect freedom. A *pulley* is a little wheel with a groove cut all round its rim, (see fig. 11,) and the string or rope passes over the wheel resting in this groove. The wheel is made capable of turning round its axis or pivots smoothly, and without resistance, so that if one end of the string has ever so small a pull exerted upon it, the pulley will immediately turn round, and allow the string to move freely in the direction in which it is pulled; or, if both ends of the string be pulled, but one side with a greater force than the other, the string will immediately move in the direction of the preponderating force, however little it may exceed the other. The use of a pulley of this kind is, first of all, to give perfect freedom of motion to the string, and secondly, to prevent the rubbing, and consequent wear and tear of the string; for if the string were passed over a pin or tack, or through a hole or ring, instead of over a pulley, it is clear that there would be a good deal of rubbing, which would both resist the motion and wear out the string. When the axis or pivots round which the pulley turns are fixed, the pulley is called a *fixed pulley*. Of movable *pulleys* we shall speak hereafter. Returning to

Fig. 10.

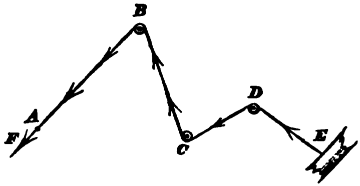


Fig. 11.

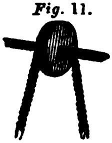


fig. 10, the string $ABCDE$ passes over the pulleys B, C, D, and is fixed to some obstacle at E. A force represented by the arrow F, pulls the string at A, and draws it tight, the effect of which is that a force of *tension* is sent along the string over the pulleys, to the obstacle at E. This force of tension acts on every part of the string, exactly in the same manner as in the case of the links of the chain above mentioned, and its direction is always *along* the string. This is represented by the arrows in the figure; the tension at B acts in the direction of the arrow at B, the tension at C acts in the direction of the arrow at C, the same may be said respecting D, and finally the tension at E pulls the obstacle in the direction of the arrow at E.

(47.) Furthermore, all these tensions are equal in energy to the original force F, so that this force is transmitted along the string over each pulley, and finally is brought into action upon the obstacle at E. Practically speaking, however, this is not strictly true, for the pulleys can never be made so exactly, and with such smooth pivots, that they are capable of turning round with perfect freedom: there must always be a certain amount of roughness and friction, as it is called, which does not allow the pulleys to turn without some degree of resistance; and the consequence of this is, that a certain small amount of force is lost or wasted at each pulley, and thus the force F is not transmitted along the string to the obstacle, without some little diminution.

(48.) What we have just stated supposes the string to be perfectly flexible, and this we may say is *true of very thin strings* made of flexible mate-

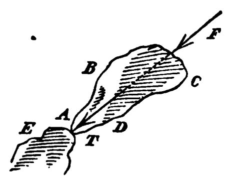
rial. With thick strings or ropes, however, the case is different; they are by no means perfectly flexible, but have always a certain degree of rigidity or stiffness, which must be allowed for in estimating their mechanical effect.

The force of tension plays an important part in Mechanics, and must be taken into account in a great number of problems which we shall give hereafter.

(49.) *The force of Thrust.*—This force, which is exercised by rigid or perfectly stiff bodies, is analogous to the force of tension, and equally important in Mechanics. When a man pushes with a stick or stiff body of any kind against an obstacle, he is said to exert a *thrust*, which is transmitted along the stick to the obstacle. A tension arises from a *pull* with a string, a thrust from a *push* with a stick.

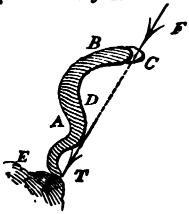
(50.) A thrust is always transmitted in the direction of the force which produces it, and in this particular it differs from a tension. Thus, let $ABCD$, (fig. 12,) be the rigid body, which is employed to transmit the thrust, F the force which produces the thrust, and E the obstacle upon which the thrust ultimately acts. Then, supposing $ABCD$ to be perfectly stiff and rigid, the force which is exerted upon the obstacle E, in virtue of the thrust transmitted through the interposed body $ABCD$, will *always* act in the direction of the force F produced, as is represented in the figure by the arrow T, *no matter what the shape of* $ABCD$ *may be.* For instance,

Fig. 12.



if *ABCD* be of the shape represented in fig. 13, the arrow *T*, (which, as in fig. 13, represents the force of thrust brought into play on the obstacle, by the action of the force *F* transmitted through the rigid body *ABCD*,) will always point and lie in the same direction as the arrow *F*, provided, of course, the body *ABCD* be perfectly stiff.

Fig. 13.



The distinguishing character, then, of a force of thrust is, that it is always transmitted through a rigid body in the same direction as that of the force which originally produces it. We may also observe that the original force is transmitted through the rigid body *undiminished*, that is, the force *T*, in any of the above figures, is always equal in energy to the force *F* which produces it.

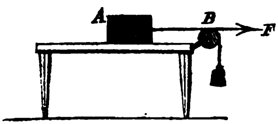
(51.) What we have just stated is an instance of a principle we shall presently employ, called the *Principle of the Transmission of Force through a Rigid Body*, which asserts, that when a force acts upon a rigid body, it is transmitted through it unchanged in direction, and undiminished in energy. The effect of a force on a fluid confined in a vessel is very different, as we shall explain at length hereafter, for it is transmitted through the fluid, not in one particular direction, but literally *in every direction*, up and down, right and left, and in this way it is brought into action on every point of the sides of the vessel which contains it.

OF THE FORCE OF FRICTION.

(52.) The *Force of Friction*, as it is called, is brought into play by the *roughness* of bodies placed in contact with each other; it is a force of great importance in practice, both because it is of great utility in some cases, and of considerable disadvantage in others. Without friction no structure could stand securely, no arch could be made use of, no nail or screw could hold framework together. At the same time, the friction in the different parts of machinery, whether stationary or locomotive, is a serious cause of waste and expense.

(53.) When a body *A*, (fig. 14,) is placed upon a flat table, we know by constant experience, that it requires a certain amount of force to make it move in any direction upon the table; this may be made evident very simply by the following method:—

Fig. 14.



Fasten a string to the body *A*, let it pass over a fixed pulley *B* at the edge of the table, and suspend a weight by it, or, what is better, a scale-pan, into which we may put any amount of weight as we please. If, now, we put a very small weight into the scale-pan, supposing the scale-pan itself to be light, we shall generally find that the body *A* will not be moved by it. But if we go on putting more weight into the scale-pan, and so increase the pull on the string, we shall find at length that *A* will begin to move. This proves the existence of a force which prevents the body *A* from moving until a certain amount of weight

pulls the string. The weight put in the scale-pan produces a force of tension on the string, which is transmitted over the pulley, and is so brought to bear in a horizontal direction on the body, as is represented by the arrow F; in fact, the string and pulley together are merely a contrivance for making the vertical force of gravity, that is, the weight in the scale-pan, act horizontally on the body A, in order to determine whether such an amount of force is sufficient or not to cause A to move.

The force F, then, which is thus made to act horizontally on the body A, does not move A at first; and the cause of this is the roughness of A and of the table, which prevents A from moving. The force F is therefore destroyed by the roughness, and this of course is effected by a force equal and opposite to F, (as we have already explained in speaking of *Resistance*,) which the roughness brings into action against F. The force thus brought into action by the roughness is called the *Force of Friction*. It is of course, in this case, a horizontal force, since it directly opposes the force F, and for the same reason it acts in a direction contrary to that in which F tends or endeavours to move the body.

(54.) It is important to remark, that the roughness is capable of bringing into play a force of friction of any magnitude, within a certain limit. This is proved by the fact, that the body A will not be moved by the string until a certain amount of weight is put in the scale-pan; but when that amount, or anything over it, pulls the string, A immediately moves. The force of friction can therefore *always destroy* the force F, whatever be

the magnitude of *F*, provided it does not exceed a certain limit. It follows, therefore, since the force of friction must be exactly equal and opposite to *F*, in order to destroy it, that the roughness may bring into play an opposing force of friction of any magnitude, under a certain limit. What that limit is we shall explain more fully hereafter.

(55.) That the roughness is the real cause of this resistance to the force *F*, may be easily proved by increasing or diminishing the roughness, when it will be found that the resisting force of friction will be increased or diminished accordingly. Thus, if the table be made of oak, with the grain in the direction of the string, and if the body *A* be 100 lbs. weight of wrought-iron, it will be found that no weight under 62 lbs. put in the scale-pan will move *A*, but any weight above 62 lbs. will move *A*. In this case, then, the resisting force may be of any amount not exceeding 62 lbs. Now, if the same experiment be tried upon a wrought-iron table, instead of an oaken one, it will be found that the greatest resisting force will be only 14 lbs. If we roughen the surfaces of the body and table with a file, the resisting force will be greatly increased. In fact, it will soon be perceived on trial, that an increase of roughness will always increase the resisting force, and enable it to sustain a greater amount of weight in the scale-pan.

The most polished surfaces will always exercise a certain amount of friction on each other, because no polish can be so perfectly effected by the mechanical means we possess, as not to leave a certain degree of roughness on the surface. The *least amount of* friction is found in the surfaces of

bodies which have a natural polish, and a considerable degree of hardness and brittleness combined, as for example, in the case of ice sliding upon ice.

(56.) One other important point may be observed here respecting the force of friction, namely, it increases with, and in proportion to the force which presses the body against the table. In the case above supposed, of iron on an oak table, the body *A* weighs 100lbs., and therefore 100lbs. is the force which presses *A* against the table. The force of friction, in this case, at greatest is 62 lbs. Now, if we put a weight of 100lbs. upon *A*, and so make the force which presses it against the table 200lbs., it will be found that as much as, but not more than 124lbs., or twice 62lbs., may be put in the scale-pan without moving *A*. And if we put a weight of 200 lbs. on *A*, and so make the whole pressure 300 lbs., we shall find that as much as, but not more than 186 lbs., or three times 62 lbs. may be put in the scale-pan without moving *A*. Whence it is evident that the force of friction increases with, and proportionally to the force which presses the body against the table.

OF ANIMAL FORCE.

(57.) The muscular strength which animals possess gives them the power of exercising at their will very important and sometimes very considerable forces; and when several combine together, the effects produced by animal effort are very stupendous, as for instance, the enormous structures we see continually erected, the harbours, canals, and railways, and such like works. But the chief *peculiarity of the* forces which animals,

and especially man, can exert, is their applicability to produce every kind of motion, however complicated, and so minister to our necessities and comforts. We can only say a few words here on the subject of muscular force.

(58.) We may first observe, that the manner in which an animal moves his limbs is by the contraction of the muscles attached to the different bones. By a wonderful stimulus, which is conveyed, at our will, by special nerves, from the brain to different parts of the body, a swelling and consequent contraction is produced in the muscle; and in this way a pull or tension is brought into play upon the bone. This will be made evident to any one who observes the manner in which he lifts his hand by a motion from the elbow. If he feels the arm between the elbow and shoulder, he will perceive the swelling of the muscle when he lifts the hand, and the fact that it pulls the bone a little way in front of the elbow-joint, and so raises the hand. In exactly the same manner, all other voluntary motions are produced in the body, by the contraction of muscles.

(59.) The simple act of walking is generally but little understood: it is effected chiefly by those muscles which throw the body forward, and which, if the feet were not moved, would upset the man. The man, resting on one foot, inclines his body forwards, but arrests his fall by putting out the other foot: then resting on that foot he again inclines his body forwards, and puts out the former foot to arrest his fall: and by continually repeating this action he performs the act of walking. The motion of walking is in fact extremely like the *rolling* of a wheel without a rim, having only

spokes: the centre of the wheel, like the centre of gravity of the man, is continually tending to fall forwards by turning round the extremity of the spoke which touches the ground, but the spoke just in front of that is continually arresting the fall. The difference is that in man the two legs, being capable of being brought rapidly forward in succession, answer instead of the great number of spokes all round the wheel.

(60.) Engineers have estimated the average amount of work which animals are capable of doing under various circumstances: we shall give a few of their statements.

An unloaded man can walk 31 miles per day on level ground; which, estimating his weight at 10 stone, and supposing that he rests for 14 hours per day, is equivalent to the work of moving 10 stone at the rate of about 3 miles per hour while he is actually walking.

A strong man carrying 128 lbs. can walk at the rate of $9\frac{1}{2}$ or nearly 10 miles per day.

A man walking up an easy flight of steps can ascend a vertical height of nearly but not quite 2 miles per day.

A man walking up the same steps and carrying a weight of 150 lbs. can ascend only a vertical height of half a mile daily.

A man drawing a boat on a canal can do 750 times more work daily than he could do in moving materials with a wheel-barrow.

A man moving materials with a wheel-barrow can do 9 times more work than when digging with a spade.

A horse carrying a moderate burden, such, for instance, *as a man, can do 10* times as much work *as a horse turning a mill.*

A dromedary carrying a man can do twice the work of a horse carrying a man.

(61.) The subject of the work done by animals of various kinds and in various circumstances is one of much practical interest, as the above statement may show. We shall give one other illustration of this subject.

Suppose that a strong man, a porter for instance, has to transport a large quantity of material from one place to another, what is the best load for him to take each time? If he takes too large a load he will go slowly, if too small a load he will waste time in going and returning too often. The answer is—he ought to take about 135 lbs. each time, supposing the ground to be level. Doing this he will walk about 7 miles per day, and transport a greater quantity of material than he could do if he took a smaller or a greater load each time.

OF ELECTRICAL AND MAGNETICAL FORCES.

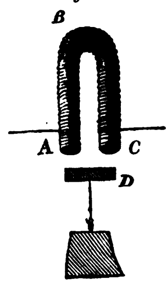
(62.) *Electrical Attraction and Repulsion.*—The electric fluid, which produces such powerful and destructive effects in the form of lightning, is capable of exercising forces of a very wonderful nature, which appear to play a very important part in the various operations and processes of the material world, especially those connected with vegetable and animal organization. The forces of attraction and repulsion which bodies exercise on each other when they are charged with electric fluid are, if we except thunder and lightning, the simplest and most familiar instances of the effects produced by this subtle agent. If a piece of sealing wax be rubbed with a piece of dry silk it *will become* charged with electric fluid, and will, *in consequence,* acquire the power of attracting

light bodies, such as small pieces of paper, thread, feathers, or the like. If a piece of glass, a glass tube or small phial for instance, be rubbed in the same way, it will exhibit like effects of attraction. But there is a curious difference between the electric fluid in the sealing wax and in the glass, which may be thus stated.

(63.) Two bodies charged with *vitreous* electricity (the word *vitreous* means "*produced from glass,*") repel each other; two bodies charged with *resinous* electricity (*resinous* means *produced from resin*, or a *resinous substance*, like sealing wax), likewise repel each other; but, if one body be charged with vitreous electricity and the other with resinous, they attract each other. From this it is supposed that there are two kinds of electrical fluid, one of which is called *vitreous* and the other *resinous*; that vitreous electricity exercises a repulsive power on vitreous electricity, but an attractive power on resinous, and that resinous electricity exercises a repulsive power on resinous electricity, but an attractive power on vitreous. These two kinds of electricity have also been called *positive* and *negative* electricities.

(64.) But the electric fluid when put in motion along spiral or twisted wires, exerts a much more curious kind of force. If we bend a rod of soft iron into the shape *A B C*, fig. 15, and twist a wire covered with silk about it, round and round from one end to the other, and then send a continual stream of electric fluid *through the wire, the piece of* iron

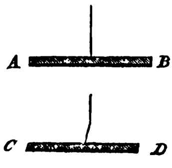
Fig. 15.



will acquire a very considerable power of attraction on any other piece of iron brought near it at *A C.* Indeed in this way, by means of a very feeble electrical current, we may produce such a power of attraction in the piece *A B C,* that if we put another piece of iron *D* underneath it, and suspend several stone weight from *D,* the attractive power in the piece *A B C* will pull *D* upwards and sustain the whole weight. The moment we stop the current of electricity through the wire, the piece *A B C* loses altogether its power of attraction, and *D* falls.

(65.) *Magnetic Attraction.* — The well-known substance called *loadstone* (an ore of iron) exercises an attractive power on iron extremely like that which we have just described, and which is attributed to what is called the *magnetic fluid,* or *magnetism;* whatever exhibits this kind of attraction being called a *magnet.* It is now proved beyond doubt that magnetism is only a modification of electricity; in fact all the phenomena attributed to magnetism may be produced by sustained currents of the electric fluid, and furthermore, the electric fluid may be drawn from a magnet. The earth is a great magnet, and probably it is so in consequence of currents of electricity circulating round it, having some connexion with its rotation about its axis. The magnetism of the earth is familiar from the curious effect it produces of making a magnetic needle point towards the north.

Fig. 16.



(66.) The nature of magnetic force may be briefly stated thus. If *A B* and *C D,* fig. 16, be two magnetic bars, whether natural

magnets, or artificial, or produced by electrical currents; and if they be suspended or balanced so as to be capable of turning freely round their middle points; then the extremities, or *poles* as they are called, of these bars have the following properties.

One extremity or pole will always point northward or nearly so, *that* extremity of the bar is called its *north pole*, and the other extremity its *south pole*.

Assuming A and C to be the north poles of the two magnets, and B and D the south poles, if A be brought near C they will repel each other, and if B be brought near D they will also repel each other: but A will attract D, and B will attract C. In other words, poles of the same name repel each other, and poles of a contrary name attract each other.

CHEMICAL AND OTHER FORCES.

(67.) We have already spoken of the forces of *Cohesion* and *Repulsion*, and therefore we need not allude to them here, more than to observe, that they are very important to be noticed in a treatise on Mechanics, as will appear hereafter. There are other forces of a similar nature called *Chemical Forces*, which produce most extraordinary effects, but of whose nature we know very little. These are the forces which give rise to those chemical mixtures and transformations of substances whereby their properties are modified and changed in the most curious and unaccountable manner. Thus the two gases called oxygen and hydrogen, when mixed together *in certain* proportions, form an *invisible aeriform substance* in which chemical

forces of tremendous power are, as it were, concealed and ready for action. For, if a lighted taper or electrical spark be applied to this mixture, a violent explosion takes place, caused by the rushing together of the particles of the two gases, in consequence of intense attractions which they exercise on each other. The result of the explosion is that the gaseous mixture is transformed into a few drops of water. Gunpowder is a familiar instance of the same sort of violent action, only instead of a gas being converted into a liquid, a few solid grains are almost instantaneously expanded into a gaseous form.

(68.) Respecting a variety of other forms which might be enumerated here we have not space to say any thing; the forces of gravity, cohesion and repulsion, resistance and friction, tension and thrust, are those we have to consider most frequently in Mechanics. These and the other forces above spoken of are the principal causes of the various phenomena of the material world, for almost all these phenomena are simply manifestations of force and consequent motion.

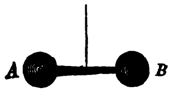
This assertion applies to many things which appear to be something different from and of a more subtle nature than mere effects of force and motion. Thus *Sound* is but a tremulous motion communicated to the ear by the vibrations of the air, and *Light* is of the same nature. All the varieties of tone, and language, and musical pitch, all the shades of colour, and the numberless effects thence resulting, are but modifications of motion, and effects of force.

EXPERIMENTAL ILLUSTRATIONS.

(69.) We shall give some experimental illustrations of what has been said with regard to the force of gravity. We may observe, first, that the existence of the force of gravity may be proved by a very remarkable experiment which we shall briefly describe.

(70.) *Cavendish's Experiment.*—This experiment was described by Cavendish in the Philosophical Transactions for 1798; its object was to show the existence of the attraction of gravitation between substances of moderate dimensions on the earth's surface, and to determine the weight of the earth. Two balls of lead, *A* and *B*, fig. 17, were fixed at the ends of a rod, and the rod was suspended by its middle point. The object of this arrangement was to allow the leaden balls to move with the greatest possible freedom, so that a very small force acting upon either of them might become immediately sensible by communicating motion to it. The balls thus suspended were put under a wooden box, to prevent any disturbance or agitation being communicated from the air. A lamp was put in the box on one side and a telescope inserted on the other, so that, by looking through the telescope, any motion of the balls might be easily observed.

Fig. 17.



Now he found that when large leaden balls were placed outside the box, near each end of the rod on which the leaden balls just spoken of were fixed, the effect was, that the extremities of the rod were *drawn towards* the large leaden balls,

and the rod being thus disturbed out of its position of rest, vibrated backwards and forwards horizontally like a pendulum.

By observing the manner in which this vibration took place, Cavendish was able to calculate how great the attraction of the large leaden balls was upon those fixed on the rod. We are not sufficiently advanced to explain the principle of this calculation here; suffice it therefore to say, that the result proved beyond doubt the fact that the leaden balls exercised an attraction of gravitation towards each other, according to the law of the inverse square of the distance above explained; but the most interesting and important deduction from the experiment was that the earth was on the average about $5\frac{1}{2}$ times heavier than water,—that is, that the weight of the whole earth was about $5\frac{1}{2}$ times more than the weight of an equal bulk of water. This makes the weight of the whole earth to be somewhat more than

1,000,000,000,000,000,000 tons,

that is a million of millions of millions of tons in round numbers. The exact numbers have been made out in this case, but it would be useless to give them.

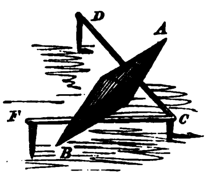
(71.) It may seem very extraordinary, and almost incredible, that the weight of the earth may be determined by watching the vibrations of leaden balls suspended as above, but the fact is so beyond doubt, and it forms a very striking illustration of what may be done by a knowledge of mechanical principles. At first sight it would appear an impossibility for such a creature as man to determine the weight of the enormous sphere on which *he dwells*, but, as we shall presently show, the

laws of mechanics, properly understood and applied, enable him to do this with comparative ease. And it is still more extraordinary that, by the same laws, he can prove that the Sun weighs 354,936 times more than the Earth, but that the material of which the Earth is composed is on the average about 4 times heavier than that of which the Sun is composed. And in the same manner the weights of the planets have been determined, as we shall show hereafter.

(72.) *Attraction of Mountains.*—From what we have stated respecting the attraction of gravity residing in every particle of matter, it follows that a great mountain must exercise some sensible amount of attraction on bodies near it. This was found to be the case by Dr. Maskelyne, (and others,) who suspended a plumb-line, and found it was drawn, though not to any extent, out of the true vertical by the attraction of neighbouring mountains. Near the mountain called Schehallien in Scotland Dr. Maskelyne tried experiments with plumb-lines north and south of the mountain, and he found, by means of astronomical observations, that the north plumb-line made an angle with the south plumb-line $11\frac{1}{2}''$ more than it would do in a plain country. This showed the attraction of the mountain, by which the plumb-lines were drawn out of the true vertical towards the mountain, and so made a greater angle with each other than they would do if there were no mountain mass to attract them.

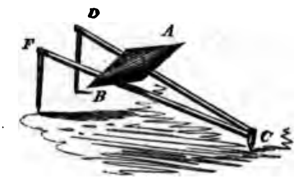
(73.) *The Centre of Gravity always descends as low as it can, but never rises.*—A very simple experiment may be mentioned which puts this principle in a striking point of view. If we get a piece of

Fig. 18.



wood or metal of the shape of a double cone, *A B*, fig. 18, and form a sort of inclined plane with two rods *C D*, *C F*, which join at *C*, making an angle with each other which may be increased or diminished at pleasure: and, if we place the double cone at the bottom of this inclined plane, as in fig. 18, it will run up the inclined plane to the top, provided the angle *D C F* be sufficiently large. But, if we make the angle *D C F* small, and put the double cone at the top of the inclined plane, as in fig. 19, it will run down the plane.

Fig. 19.



All this is easily explained, for it will be perceived, that, when the angle *D C F* is sufficiently large, the central point of the double cone, which is the centre of gravity, really descends when the cone ascends up the plane, and that at the top of the plane the centre of gravity is actually in a lower position than it is at the bottom. Whence, the cone must run up the plane in order that the centre of gravity may descend as low as it can, according to the principle we are now considering. In the case where the angle *D C F* is small, it will be perceived that the centre of gravity ascends when the cone runs up the plane, and therefore the cone must run down.

(74.) We shall not now delay to give illustrations of the other forces we have spoken of in the *present chapter*, as many opportunities of doing

so, and with greater advantage, will occur in different parts of this treatise. Enough has now been said, by way of introduction, respecting matter and force, to make the student who has never thought upon the subject familiar with the idea of force, and, to a certain extent, with the different kinds of forces in nature, and their effects on material substances.

IMPORTANCE OF MECHANICS AS A BRANCH OF STUDY.

(75.) The object of Mechanics being, as we have stated, to investigate the effects produced by forces upon material substances, it is evident, that the greater part of those sciences which treat of the phenomena of the material world must depend more or less upon Mechanics, and that without a complete knowledge of the laws and principles of Mechanics, it is impossible to make any great progress in those sciences. Thus the motions of the planets in their orbits are effects resulting from the force of gravitation, and therefore, to understand the motions of the planets completely we must know how to determine and trace out the effects of such a force, which of course can only be done by the help of Mechanics. In like manner, sound is a vibratory motion resulting from the elastic force of the air, and therefore, in investigating the laws of sound we must be familiar with the principles of Mechanics. But it is not necessary to multiply examples; the above enumeration of the different kinds of forces in nature is sufficient to prove the importance of that science *in which force and the* effects of force form

the subject of investigation. The great variety and extent of the phenomena which are produced by these forces directly and indirectly may well induce any one who wishes to study the laws of nature to make himself well acquainted with the rules of Mechanics and their application.

(76.) The study of Mechanics is also important as regards the common affairs of life, especially to those who may be engaged in any of the practical pursuits where the power of reasoning upon and estimating the effects of force in its different modifications is valuable and in many cases necessary. In our days it is scarcely necessary to insist on this point, when so much is effected by mechanical means, and so much has been gained by the improvement of machinery. Whether it be in building a house, or making a road, or constructing drains, or supplying water, or in the more stupendous works of engineering science, or in the operations and processes of manufacturing art, in all an accurate and extensive acquaintance with the science of Mechanics is invaluable.

(77.) If it were only to discourage the absurd use which many ignorant persons make of their inventive powers, the study of Mechanics would be very useful. It is really surprising how much time and money have been spent in inventing and constructing machines which can never answer the purpose intended, and in which, notwithstanding, the inventor feels the most extraordinary amount of confidence, from ignorance of the commonest mechanical laws. The author of this treatise has at various times been consulted on the subject of inventions the supposed value and merit of which consisted in what was really a contradic-

tion to a well-established law of Mechanics which states, *that Motion can never generate Force.* In one case more than 5,000*l.* had been spent in constructing a very complicated machine, which was to have superseded steam-engines in the production of moving power, but which, in the end, could not produce power enough even to move itself. In another case a monster wheel of 40 feet diameter, moved by a horse, was to help the horse, simply by its rotation, in such a wonderful way, as to enable him to do the work of 10 horses. There have been an incredible number of inventions for producing perpetual motion, all of which have grievously disappointed the inventors. A great number of patents have been taken out, at an average cost, we suppose, of 300*l.* or so, for what are called rotatory steam-engines, which have never been brought into use. So that, taking the waste of time, ingenuity, and money in constructing these useless machines, to which in many cases the expense of the patent must be added, the loss has been considerable indeed.

Now in all these cases a competent knowledge of Mechanics would have saved the poor inventor from disappointment, and never allowed him to indulge in the delusive hope of doing that which the laws of nature forbid. By a knowledge of Mechanics the absurdity of making motion generate force would have been confuted, and men would not have constructed machines to pump up water, hoping and believing that the water by falling down again would produce force enough, not only to keep the pump going for ever without requiring any other external help, but also would furnish a *large surplus of* power to be applied to

move machinery, or do other work.* A knowledge of Mechanics enables a man in a great majority of cases to estimate beforehand the effect which any combination of mechanical powers or other contrivances will produce, and by so doing an inventor may tell the value of his invention, and calculate his chance of success, by simple operations of arithmetic, without being put to the expense and trouble of actually constructing the machine. Had this been done in the case of nearly all the rotatory steam-engines which have been constructed, the inventors would have found out *in a few hours or days*, what they were afterwards taught by *years* of trouble and expense ending in disappointment, that the fancied superiority of their invention over the common steam-engine was a delusion.

(78.) But the chief utility of the study of Mechanics to many persons is the excellent exercise which it affords to the mental powers. It is a subject in which we have a combination of *deductive* and *inductive* reasoning, of strict demonstration and appeal to observation and experiment. Some of the fundamental laws of Mechanics are proved by the *inductive* method,—that is, the method by which a law or principle is established from a great number of instances or examples in which it is proved to be true by actual observation or experiment. Again, a great number of propositions in Mechanics are *deduced* by such demonstrative reasoning as we find in Euclid or any other part of abstract mathematics, with this

* The number of hydraulic machines for perpetual motion, all based upon the erroneous principle here stated, that have been invented, is considerable.

difference, however, that the process of demonstration in Mechanics is generally not only shorter and simpler, but also more intelligible to minds unpractised in such exact reasoning, because the thing to be proved is more obvious and common place, and capable of very palpable illustration from ordinary experience.

Now to confine the mind to purely abstract and deductive reasoning is naturally enough considered by practical men as prejudicial to its full development; and the same may be said of the opposite system of neglecting that training in strict demonstration which mathematics furnishes, and teaching nothing but sciences of observation and experiment. In Mechanics, however, and the numerous applications of mechanical science, we have a valuable combination of both systems, the deductive and the inductive, and it is on this account, independently of any practical utility, that we look upon the mechanical sciences as forming together a branch of instruction better calculated to improve the mind than any other.

(79.) Lastly, the mechanical sciences are necessary to the proper understanding, not only of some of the most stupendous proofs of design and unity in the constitution of the universe that we can appeal to, but also of many other evidences on a smaller scale of Creative skill, in the various mechanical contrivances we meet with in our own bodies and in those of animals, and in other parts of the material world. Into whatever part of nature we look, whether animated or not, whether within the confines of this earth, or beyond it in the vast and innumerable systems of suns and planets, we *find everywhere* mechanical proofs of *Divine agency and wisdom.*

CHAPTER III.

MEASUREMENT AND GRAPHICAL REPRESENTATION OF FORCES—DIVISION OF THE MECHANICAL SCIENCES.

MEASUREMENT OF FORCES.

(80.) *Measurement, what?*—The *measurement* of a quantity of any kind, whether it be length, solidity, weight, time, or the like, consists in estimating numerically the magnitude of that quantity as compared with a standard, or *unit*, as it is called. We measure, for instance, a distance by comparing its length with a foot, supposing a foot to be our unit, and by expressing the result of that comparison by a number. The comparison in this case is effected by trying how often a foot is contained in the distance. In other cases the mode of comparison is different.

In measuring a quantity, then, two things are necessary; 1st, to fix upon a unit or standard of reference; 2dly, to find some method of comparing the quantity to be measured with the unit, so as to determine and express by a number how much greater or less than the unit the quantity is. We shall now explain how these two things are done in the case of forces.

(81.) *Unit of force.*—In the same way that a foot, or an inch, or a mile, or any other distance may be chosen as a unit of length, so any amount

of force may be fixed upon as a unit of force. The most usual unit of force in this country, and one which we shall always exclusively employ in this treatise, except when the contrary is specified, is the *Pound weight Avoirdupois.* This is not far from the weight of a pint of water, according to the old saying,

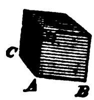
> " A pint's a pound
> All the world round."

More correctly we may say,*

> " A pint of pure water
> Is a pound and a quarter."

The number of pounds avoirdupois in a cubic foot of water is $62\frac{1}{3}$ lbs., or, more accurately, 62.3210606 lbs. A square block, as in fig. 20, measuring a foot each way, that is, *A B*, *A C*, *A D*, &c. being each a foot, is called a cubic foot.

Fig. 20.



Having settled what a pound weight is, we define the unit of force to be, that force which a pound weight exerts in pulling directly downwards. In other words, if we hang a pound weight by a string, the force acting on the string is our standard force, to which we shall refer all other forces for the purpose of measuring them.

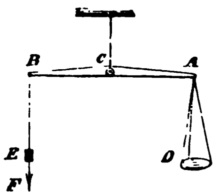
(82.)—*Methods of comparing forces with the unit of force.* The simplest method is by means of the common *Balance,* which is used to measure any forces acting *vertically dewnwards,* such as the weights of bodies. When the force does not act vertically downwards, some other method must be

* *Penny Cyclop: Art.* Weights and Measures.

employed: what is called a *Spring Balance* may be used in such a case, or a *Bent Lever Balance.* There are several methods of measuring forces which it will not be necessary to allude to here; the following are sufficient for our purpose.

(83.) *Measurement of forces by the Common Balance.* The Common Balance consists of a rod or beam *A B*, fig. 21, suspended from a point *C*, which is exactly half way between *A* and *B*, and a little over the centre of gravity of the beam, in order that the beam may rest in a horizontal position, and, if disturbed from that position, tend to return to it again. This, it will be found, will be the case if the centre of gravity be a short way below the point of suspension *C* in a vertical line when *A B* is in a horizontal position. On this point we shall speak more definitely hereafter. *C A* and *C B* are called the two arms of the balance; they are exactly equal and similar in length, weight, and form. From *A* and *B* are suspended strings which support scale-pans; we shall suppose that one of these scale-pans is removed, namely, that suspended from *A*, and that the other is retained, as shown by *D* in the figure; to make up for the removed scale-pan we must suppose a counterpoise *E* equal to the scale-pan *D* to be suspended from *A* by a string.

Fig. 21.



Now we may assume the following principle as self-evident; namely, if two equal forces act vertically downwards, one at *A* and the other at *B*, *they will balance* each other, as it is said,—that is,

they will produce no motion in the beam *A B*, inasmuch as the tendency of the force at *A* to draw down the arm *C A*, is exactly equal and directly opposed to the tendency of the force at *B* to draw down *C B*. But if one of the forces be greater than the other, the greater force will prevail and bring down the end of the beam on which it acts.

This being assumed, we may measure any force acting vertically downwards as follows. Represent the force in question by the arrow *F*, and suppose it to act on the counterpoise *E*; put several pound weights into the scale-pan *D*, and go on adding more, or taking away some if necessary, until the force *F* is balanced by them: then, by the principle just stated, the force *F* must be equal to the force exerted downwards by all these pound weights. Thus it is easy to see that *F* is measured by counting the number of pound weights which balance it. If, for instance, 10 pound weights balance *F*, *F* must be a force 10 times greater than the unit, and then *F* is properly measured and represented by the number 10. When therefore we say that a force is 10 or 20, or any other number of pounds, we mean that it may be balanced, in the manner just described, by so many pound weights put in the scale-pan.

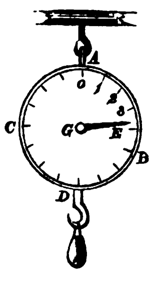
(84.) We may frequently find, that no exact number of pound weights will balance *F*; in such a case we must use subdivisions of a pound. Suppose, for instance, we have weights equal to 1-10th of a pound, and others equal to 1-100th of a pound, and so on. Then if 15 pound weights, 7 of the weights equal to 1-10th of a pound, and 3 of those equal to 1-100th of a pound, be put in the *scale-pan, and found to* balance *F*, *F* must be

equal to the force exerted downwards by all these weights, and is therefore properly measured and represented by the sum of the numbers 15, 7-10ths and 3-100ths, or by the mixed decimal 15.73.

(85.) If it be found impossible to balance *F* by any set of pound weights and subdivisions thereof, a case possible in theory, but not in practice, then the force is said to be *incommensurable*,—that is, incapable of being measured by any number of pounds or subdivisions thereof. We shall not however trouble the student by the introduction of incommensurable forces in this treatise.

(86.) *Measurement by the Spring Balance.*—A Spring Balance generally consists of a circular box, *A B C*, fig. 22, inside which there is a strong spring fixed; with this spring the projecting hook *D* is connected, so that the spring is bent when *D* is pulled by any force. To show how much the spring is bent there is a hand *G E* capable of turning round the centre *G*, and so connected with the spring, that the more the spring is bent, the more the hand *E* moves away from its natural position, which we shall suppose to be at *A*. Round the circumference which the extremity of the hand describes, are the numbers, 0, 1, 2, 3, &c., which are put in their proper places as follows:—

Fig. 22.



The number 0 is put at *A*. A pound weight is suspended by the hook, and at the place to which the hand points there is put the number 1. Again, 2 pounds are suspended, and at the place *to which the* hand points is put the number 2.

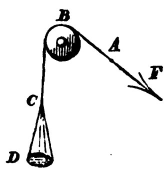
In like manner the places of the numbers 3, 4, 5, &c. are determined by suspending the corresponding number of pounds. Of course the box *A B C* is suspended from some fixed hook, or otherwise supported while these operations are going on.

From this description it is obvious that we may measure any force by means of this instrument; for, if we apply it to pull the hook *D*, and observe to what number the hand points, that number shows how many pounds the force is equivalent to.

As we may hold or support the box firmly in any position we please, and so make the projecting hook lie in any required direction, this is a convenient way of measuring forces which do not act vertically downwards. The spring balance is, however, by no means capable of the same exactness as the common one.

(87.) We might conceive forces to be measured by means of a pulley in the following manner. Suppose the arrow *F*, fig. 23, to represent the force to be measured; let it be made to pull the string *A B C* which passes over a fixed pulley at *B*, and let a scale-pan *D*, of a known weight, be suspended from the string at *C*. Then, if we put weights into the scale-pan just sufficient to overcome the force *F* and prevent it from pulling up the scale pan, the whole amount of these weights, together with the weight of the scale-pan, will be equivalent to the force *F*, and thus the magnitude of *F* will be determined. This method appears to be simple enough, but in practice it would be very inaccurate, on account

Fig. 23.



of the friction of the pivots of the pulley, and for other reasons.

(88.) *Magnitude of a force.*—From what has been said the meaning of the term *magnitude*, as applied to *force*, may be understood. By the magnitude of a force we shall always mean, the *number of pounds* to which that force is equivalent, or, what is the same thing, the number of pounds by which that force may be balanced and measured according to any of the methods just described.

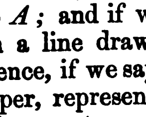
REPRESENTATION OF FORCES.

(89.) There are three things by which a force is determined, namely, its *point of application*, its *direction*, and its *magnitude*; in other words, if we are required to specify what such and such a force is, we answer by saying, that it is a force of such and such a magnitude, applied to or acting at such and such a point, and tending to move it in such and such a direction. By the *point of application* we mean of course the point to which the force is *applied*, or at which it acts; and by the *direction* we mean the *line* along which the force tends to move the point of application. Thus when we pull a body by a string, the point of application of the force we exert on the body by means of the string, is that point of the body to which the string is fastened, and the direction of the force is along the string. We have already fully explained what the magnitude of a force is, and how it is measured.

Now it is necessary, in considering and explaining the effects of forces, to have some method of representing them to the eye upon paper, so as to show the point of application, direction, and

magnitude of each force. This may be very conveniently done by means of straight lines; for a straight line, like a force, is drawn from a particular point, in a particular direction, and of a particular length. If therefore we take a certain point on the paper, to show the point of application of the force we wish to represent, and draw from that point a line in the direction of the force, it is evident that so far the force is represented by the line. It only remains to draw the line of such a length as to represent the magnitude of the force also. This is done by fixing upon some convenient *unit of length*, suppose for instance an *inch*, and considering that every unit of length in the line represents a unit of force, that is, that every inch represents a pound. Thus, if it be required to represent a force of six pounds, acting upon a point A, fig. 24, at an angle of 60° to the horizon upwards, we draw a line $A B$ six inches long, making angle $B A C$ equal to 60°, supposing the line $A C$ to represent the horizontal direction on the paper.

Fig. 24.



(90.) Whenever we speak of a line, $A B$, in the following treatise, we shall always mean, a line drawn from A to B, not from B to A; and if we speak of a line $B A$, we shall mean a line drawn from B to A, not from A to B. Hence, if we say, that any line $A B$, drawn upon paper, represents a force, our meaning is that the force acts upon or tends to move the point A in the direction $A B$, and that there are as many pounds in the force as there are inches in $A B$. $B A$ similarly would represent an opposite force applied at B.

(91.) We have here spoken of an inch as the unit of

length, but any other convenient unit will of course answer as well, if it be specified or understood beforehand what it is to be. The unit determined upon will of course depend in a certain degree upon the magnitude of the forces to be represented, and the size of the paper on which the lines are drawn. We may take as our unit an inch, half an inch, quarter of an inch, 1-10th of an inch, or any other length, as the case may be; only, the scale on which our figures are drawn should not be too small; for, if it be, we cannot expect to avoid serious errors in the measurement of the lines.

This *graphical* method of representing forces is of the greatest possible advantage in the solution of mechanical problems, and in the proof of mechanical theorems; for it enables us to show on paper the forces that are in action in any particular case, each completely represented in every particular by a distinct line; and thus the mind is assisted by the eye in considering and reasoning upon the effects of these forces. Furthermore, by drawing our figures with care and accuracy, we may, by actual measurement, make out the magnitudes or directions of unknown forces represented on the paper, and so solve a great variety of problems, as we shall show, with sufficient accuracy even for practical purposes.

(92.) This *graphical method* of solving mechanical problems by actual measurement is one which we shall often refer to. It has the great advantage of requiring no previous knowledge of technical mathematics on the part of the student. A person unacquainted even with Euclid may by *this* method solve very difficult and important

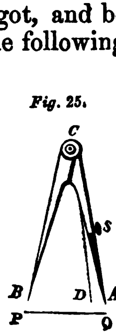
mechanical problems. There is also the method of calculating the lines and angles (in the figures representing a set of forces) by means of the rules of Geometry, Algebra, and Trigonometry. This, which we may call the *mathematical method*, requires of course a knowledge of mathematics to some extent, and is so far not suitable to a treatise like the present. We shall always employ the graphical method, without however neglecting to allude occasionally to the mathematical method, where it is possible to do so without introducing complicated or difficult mathematical processes.

INSTRUMENTS REQUIRED FOR THE GRAPHICAL METHOD.

(93.) The required instruments are few and simple. A good flat drawing-board, which need not be large, must be procured, and the paper, having been previously damped with a sponge, must be fastened on the board in the usual way, so that, when it dries, it may by its contraction be stretched quite flat and tight upon the board. A hard drawing-pencil must also be got, and be pointed finely. In addition to these the following instruments are necessary:

(94.) *Hair Compasses.*—These are like common compasses, only one of the legs is capable of being moved by a screw. They are shown in fig. 25, where S is the screw, by the motion of which the leg CA may be moved as far as the position shown by the dotted line CD. To measure a distance PQ on paper with these com-

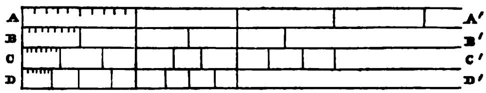
Fig. 25.



passes, put the point B of the fixed leg at one extremity P of the distance, opening the compasses so that the other point A may fall as nearly as possible upon the other extremity Q of the distance. Then, by turning the screw S, move the point A till it comes exactly to coincide with Q. In this way the compasses may be opened so as to measure, or, as it is said, *take* the distance $P\,Q$ with considerable accuracy. The use of the screw is obvious; for, without it, it is not easy, with a compass having a joint as stiff as it ought to be, to open the legs so as to measure any required distance with exactness.

(95.) *Scales.*—It is necessary to have some divided scales of equal parts of different lengths, in order to measure or lay down distances on paper. Fig. 26 represents the simplest description of

Fig. 26.

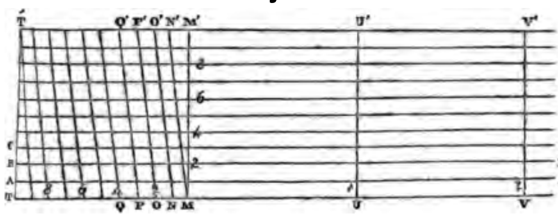


scales of equal parts. The divisions in the line $A\,A'$ are each an inch, or any other convenient unit; and one of these divisions, namely, that at A, is subdivided into ten (or sometimes twelve) equal parts. The divisions in the line $B\,B'$ are each one-half of an inch; those in $C\,C'$ one-third of an inch; and so on. The first division in each line is subdivided into ten parts. The use of these scales is obvious: if we wish to measure any distance on paper, we have only to open the compasses to that distance, as above explained, *and then,* placing the points of the compasses on

each of the lines AA', BB', CC', &c. in succession, till sufficient accuracy is attained, we may find how many equal parts, and tenths of the same, are contained in the distance to be measured.

(96.) *Diagonal Scales.*—The *diagonal scale* is represented in fig. 27, its name being derived from the lines MN', NO', &c. at the extremity,

Fig. 27.



drawn *diagonally* across the *longitudinal* lines AA', BB', CC', &c. The scale is divided into equal parts by the *perpendicular* lines UU', VV', &c. There are ten longitudinal lines AA', BB', CC', drawn at equal distances from each other. The division MT is divided into ten equal parts, MN, NO, OP, &c. The division $M'T'$ is also divided into ten equal parts $M'N'$, $N'O'$, $O'P'$, &c.; and the diagonal lines MN', NO', OP', &c. are drawn. The different divisions and subdivisions, and the longitudinal lines, are numbered as shown in the figure; but, because the numbers would come too close together if all put down, only every second number (2, 4, 6, 8,) is engraved on the scale.

The principle of this kind of scale is this, that the *diagonal line MN' cuts* off, between it and the

perpendicular line UU', small portions of the longitudinal lines, which small portions are respectively 1-10th, 2-10ths, 3-10ths, 4-10ths, &c. &c. of the division MM'. Also, the portions of the longitudinal lines intercepted between the successive diagonal lines are equal to the division MM'.

In using this scale, the two points of the compasses must always be placed on the same longitudinal line: we must never put one point on one longitudinal line, and the other point on another. This being understood, suppose one point is on the *perpendicular* line *numbered* 4, and the other point on the *diagonal* line *numbered* 7; also that both points are on the *longitudinal* line *numbered* 6. Then, considering the division MM' as unity, it is clear, from what has been said, that the distance between the two points of the compasses contains four of the large divisions, (each equal to ten times MM',) seven of the divisions equal to MM', and 6-10ths of MM'; that is, altogether, $47\frac{6}{10}$, or 47.6. In like manner, if the points of the compasses were placed, one upon the perpendicular line numbered 9, one upon the diagonal line numbered 3, and both upon the longitudinal line numbered 8, the distance measured would be 93.8, or $93\frac{8}{10}$. If we took the divisions between the perpendicular lines to be unity, the distances in these two examples would have been 4.76 and 9.38 respectively.

Hence, in measuring with this scale, we have only to remember the following rules, (in which we suppose the divisions between the diagonal lines to be unity,) namely,—

The number of the *perpendicular* line shows *tens.*

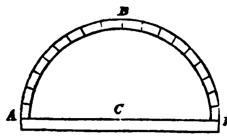
The number of the *diagonal* line shows *units.*

The number of the *longitudinal* line shows *tenths.*

Diagonal scales, well made, are extremely convenient, and sufficiently exact in the solution of mechanical problems.

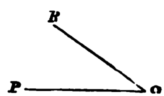
(97.) *The Protractor.*—This is an instrument for measuring and laying down angles on paper, being, in fact, an *angular* scale of equal parts. It is made of different forms; the simplest and commonest being that of a brass semicircle ABD, fig. 28. The centre C is marked by a small notch; the circumference ADB is divided into 180 degrees, which are numbered; at least, every tenth degree is numbered. If the instrument is large enough, these degrees are subdivided into two, or three, or five equal parts, as the case may be.

Fig. 28.



To measure any angle, PQR, fig. 29, put the protractor flat on the paper, making the diameter AD coincident with the line QP, and the centre C with the angular point Q. Then look where the line QR meets the graduated circumference ADB, and the number of degrees between the two lines QP and QR will be seen. If, for instance, the line QR meets the graduated circumference at the division numbered 27, the number of degrees between the two lines will be 27, and therefore the angle PQR will be $27°$.

Fig. 29.



We have not space to describe the other kinds of protractor, some of which are extremely convenient and accurate.

(98.) *Other Instruments.*—*Parallel Rulers* for drawing parallel lines, and *T Squares* for drawing perpendiculars, may be useful in a number of cases, but they are not absolutely necessary. A triangular ruler of wood or metal *A B C*, having a right angle at *A*, is very useful in laying down right angles, and drawing perpendiculars, and ought to be procured for the solution of mechanical problems, in which it is very often necessary to lay down right angles, and draw perpendiculars. The method of using this ruler is obvious: if we lay it on the paper, making *A B* coincide with any particular line, and then with the pencil draw another line along *A C*, the two lines will be at right angles to each other.

Of course, a good flat *ruler*, for drawing straight lines, is necessary in almost every case. With a good drawing-board, on which the paper is properly strained, a straight flat ruler, a pair of hair compasses, a diagonal scale, a moderate sized protracter, accurately divided, and a triangular ruler, it is wonderful what difficult mechanical problems may be solved, with considerable exactness, by the *graphical* method. Engineers and practical men often employ this method in preference to the *mathematical*.

DIVISION OF THE MECHANICAL SCIENCES.

(99.) The fundamental *Mechanical Sciences* are, *Statics*, *Dynamics*, *Hydrostatics*, including *Pneumatics*; *Hydro-dynamics*, including *Hydraulics*.

These might be very conveniently divided into two general classes, one of which might be called *Mechanics*, and the other *Hydro-Mechanics*. Mechanics would include Statics and Dynamics, and Hydro-Mechanics would include Hydrostatics, and Hydro-Dynamics. We shall say a few words in a general way respecting each of these sciences.

(100.) *Statics.*—Forces acting in combination often produce no motion, in which case they are said to *balance* each other, or to *destroy* each other, or to keep each other in *equilibrium*. The word *equilibrium* means a *poising* or *balancing* of forces, being derived from the Latin word signifying a balance, or pair of scales, combined with the word *equi*, denoting equality. That part of mechanics which treats of the equilibrium, or balancing of forces, is called *Statics*, from the Greek word signifying to *stand still*. All cases in which we investigate the effects of forces *without reference to motion*, come under the head of Statics. Thus, those propositions in which we find the single force to which a set of other forces, acting together, are equivalent, belong to statics; and a variety of other investigations, in which there is no equilibrium or balancing supposed, belong to statics, because there is no reference to motion in them.

(101.) *Dynamics.*—This word is derived from the Greek, signifying force or power; it is by no means a good and distinctive term, as it is applied at present, because it really applies to every case where force is considered. It is, however, used to denote that part of mechanics in which the various kinds of *motion* produced by forces are considered, with a view to determine what motion will *result from the action* of given forces, and,

vice versâ, what are the forces by which a specified motion is produced.

(102.) There is a very important part of mechanics in which motion is considered geometrically, as it were, without any reference whatever to force. This includes everything relating to the various forms and movements of machines, the construction of the teeth of wheels, of link work, clock work, &c. Professor Willis of Cambridge has written a valuable treatise on this part of mechanics, which he designates by the title, *The Principles of Mechanism.* Another name has been proposed, derived from the Greek word signifying *to move*, namely, *Kinematics.*

(103.) *Hydrostatics, including Pneumatics.*—The first part, *Hydro*, of this word, is derived from the Greek, signifying *water;* the second part, *Statics*, we have already explained. Hydrostatics, therefore, means that science in which forces *acting upon water and other fluids* are supposed to *balance* each other. *Air* and *gases* are here considered as *fluids*, though it was formerly usual to separate all investigations relating to the action of forces on air and gases from hydrostatics, and include them together in the science called *Pneumatics*, derived from the Greek signifying *wind* or *air.*

(104.) *Hydro-dynamics, including Hydraulics.*—*Hydro*, in combination with *Dynamics*, has the same signification as when in combination with *Statics*. *Hydro-dynamics* therefore, is that science in which forces acting upon water, and other fluids, are supposed *to produce motion. Hydraulics* relates to the mechanism of *water works* of every kind, such as pipes, pumps, water-wheels, &c. It might be, perhaps, properly called Hydro-*kinematics*, or

the science relating to the *motion* of fluids *without reference to force.*

(105.) These are the fundamental mechanical sciences, upon which so much depends in all other sciences, both practical and theoretical. We might add another mechanical science to these,—namely, that which treats of *vibratory motion* and *waves;* for everything relating to this kind of motion is now assuming great importance and interest, in consequence of the theories of *sound, light,* and *heat,* in which a great number of phenomena are evidently proved to be effects and modifications of vibratory motion.

(106.) We shall divide the mechanical sciences into three, namely, *Statics, Dynamics,* and *Hydro-mechanics.* In Dynamics we shall include *Kinematics,* or the *Principles of Mechanism.* In Hydro-mechanics we shall include *Hydrostatics, Pneumatics, Hydro-dynamics,* and *Hydraulics.* Our space, however, and the nature of the instruction we wish to convey, will prevent us from entering much into Kinematics and Hydro-dynamics; in fact, as far as these sciences are concerned, we shall only allude to points of special practical interest.

In nearly all the investigations and propositions which follow, the student is supposed to be unacquainted with technical mathematics. When anything mathematical is introduced, it will be done in such a way as not to interfere with the progress of the non-mathematical student.

PART II.—STATICS.

CHAPTER I.

PRELIMINARY STATEMENTS.

(1.) In commencing a scientific treatise it is generally necessary to give some preliminary statement respecting the terms and principles afterwards to be made use of. This includes in most cases, certain *definitions*, *axioms*, and *principles*, upon which all the propositions and investigations of the treatise are made to depend. A *definition* is a short, but exact explanation of the meaning of any term that is to be made use of. An *axiom* is a self-evident truth stated, formally, for the purpose of being appealed to in future demonstrations. A *principle* is a truth of the same nature as an axiom, or nearly so, which, from its great importance in subsequent reasonings, deserves to be brought prominently forward, and called by a special name. In many cases, a *principle* is not quite self-evident, and requires to be proved, though always the proof is very simple. When the proof is derived from experiment or

observation, but not from mere abstract reasoning, the principle is said to be an *experimental* or *physical principle.* Axioms also, though really self-evident, often require something more than a mere statement to make their truth obvious. We shall now state and explain those *definitions*, *axioms*, and *principles*, which are required in commencing the study of Statics, reserving all that are not immediately necessary for the occasion on which they may be wanted.

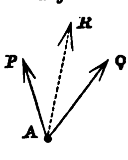
Many of the following definitions and axioms relate to what is called a *resultant*, which is a term of great importance in mechanics generally; indeed, a considerable part of Statics consists in rules for finding the resultant of forces in various circumstances. The other definitions and axioms chiefly relate to the *balancing* or *equilibrium* of forces.

STATICAL DEFINITIONS.

(2.) *Resultant.*—When two or more forces acting together produce the same effect as a single force, that single force is said to be their *resultant:* that is, the force which *results* from, and exhibits the effect of the joint action of two or more forces, is called the resultant of those forces.

Thus, if P and Q, (fig. 30,) be two forces acting at the same time upon a particle A, they must tend to move A in some particular direction, which shows that they produce the same effect as a *single* force; for the effect of a single force is to produce a tendency to motion in some particular direction. What that direction is, we shall hereafter determine; it is

Fig. 30.



sufficient now to observe, that it must lie *somewhere between* the directions of the two forces. For, if Q alone acted, the particle would move in the direction of Q, and therefore, when P acts along with Q, the effect must be to move the particle somewhat to the left of the direction of Q; in like manner, if P alone acted, the particle would move in the direction of P, and therefore, when Q acts along with P, the effect must be to move the particle to the *right* of P. It appears, then, that when P and Q act together, the particle must move in a direction to the right of P, and to the left of Q; that is, in a direction *between* P and Q.

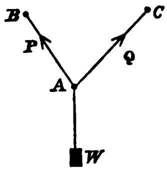
We have represented the unknown direction in which the particle will move, when P and Q act together upon it, by the dotted arrow R. The effect of the joint action of P and Q is then the same as that of some single force, acting in the direction R; that force is the resultant of P and Q.

(3.) If more than two forces acted upon the particle, we might be shown, in a similar manner, that they produce, by their joint action, the same effect as a single force: that single force is their resultant. When two or more forces act, not on a single particle, but at different points of a body, they do not always produce the same effect as a single force, as will presently appear. In such a case there is no resultant.

(4.) There are many familiar instances of two or more forces producing the same effect as a single force, of which we may advantageously mention a few here. A common kite is acted on by the pull of the string and the force of the wind; these two forces together produce the effect of a single upward force, equal to the weight of

the kite, and the result is, therefore, that the kite is supported in the air. The pressure of the wind upon the sails of a ship, and the resistance of the water against the rudder, keel, and side of it, are two forces, which make the ship move in a direction different to that of either of them. If we hang a weight W by two strings, AB, AC, (fig. 31,) the pulling forces or tensions exerted by the strings, which are represented by the arrows P and Q, produce the same effect as a single upward force equal to the weight W, and thus W is supported.

Fig. 31.



(5.) *Composition and Resolution of Forces.*—A set of forces, whose joint action is estimated, and resultant determined, are said to be *compounded into their resultant.* The process of finding the resultant of a set of forces, is called the *composition* of those forces. With reference to this mode of speaking, the forces are said to be the *components* of their *resultant.* These terms are of constant use in Statics.

The process of finding the components, whose joint action produces a certain resultant, is called the *resolution* of forces. A resultant is said to be *resolved* when its components are found.

Thus, referring to fig. 30, art. 2, the forces P and Q are the *components* of the force R; and if, in any investigation, we were to substitute R in place of P and Q, as we might do, that would be called *compounding* the forces P and Q into their resultant. On the other hand, if we were to substitute P and Q in place of R, that would be called *resolving* R *into* its components.

(6.) *Rigidity.*—Particles are said to be *rigidly connected*, or, in other words, to form a *rigid body*, when it is impossible to alter the distances between them by any amount of force. If, therefore, two or more particles be rigidly connected, they cannot be made to approach towards, or recede from each other, by forces of any magnitude.

No body in nature is absolutely rigid, but, for the purposes of demonstration, the existence of such bodies may be supposed, in the same way that the existence of a perfectly straight line without length or breadth, is supposed in geometry, though it is not possible to draw such a line. Practically speaking, a body is said to be rigid, as we have observed in a former chapter, when no ordinary amount of force can alter the distances between its particles, or make them approach towards, or recede from each other.

(7.) *Freedom and Constraint.*—A particle is said to be *free* when there is nothing to prevent its moving in *any direction*, or, in other words, when the *least* amount of force acting upon it *in any direction*, will move it.

A particle is said to be *constrained* when there is some particular direction in which it cannot move, or some other kind of limitation to its motion.

Thus, if a particle P, (fig. 32,) be suspended by a string PA from a fixed point A, it may move to the right or to the left, or upwards, but it cannot move downwards; it is *constrained* to move any way but downwards.

Fig.32.

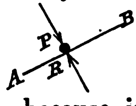
Again, a particle in a tube, in which it fits, like a bullet in a gun, is capable of moving one way or the other along the tube, but it cannot move at right angles to the tube; it is *constrained*

therefore to move along that particular line or curve which the tube forms, and in no other direction. Again, if a particle be placed upon a plane, whether it be a horizontal or inclined plane, it may move freely in any direction along that plane, or upwards, but it cannot penetrate through the plane. It is *constrained* therefore, so that it cannot move in one particular direction, namely, perpendicularly through the plane, but it may move in any other way.

(8.) *The Reaction of Constraint.*—We have already stated that a *reaction* is a force of resistance brought into action by another force. When a particle is pressed against an obstacle by any force, the obstacle *returns* the pressure, and that *return pressure* is called the *reaction* of the obstacle. In all cases of constraint there must be a resisting force brought into play, in a greater or less degree, and that force may be called the *reaction of constraint.*

(9.) Thus, in fig. 32, which represents a particle P suspended by a string PA from a fixed point A, the weight of the particle tends to pull it down, but the string resists this tendency by exerting an upward pull or tension; this upward pull of the string is a force called into action by the weight of the particle pulling it in a direction in which it cannot move; that is, there is a reaction of constraint exercised by the string in this case.

Fig. 33.



(10.) Again, suppose that a particle is placed upon a plane represented by the line AB, and suppose any forces act upon it and press it with a force P against the plane, then the plane, because it does not suffer the particle to move through it,

will resist that pressure, and the force of resistance R thus brought into action against the force P is, in this case, the reaction of constraint.

(11.) *Direction in which the Reaction of Constraint acts.*—When a particle is constrained, it is prevented from moving in a particular direction, and therefore the force of resistance which so constrains or limits its motion, must act in the contrary direction. The reaction of constraint, therefore, always acts in the contrary direction to that in which the particle is not allowed to move by the constraint.

(12.) Thus, in the case supposed in Article 9, the particle is not allowed to move downwards, and therefore the tension of the string, which is the reaction of constraint, must act upwards. Again, in Article 10, the particle is not allowed to move in the direction P at right angles to the plane, and therefore the resisting reaction of constraint acts in the direction R, which is also at right angles to the plane, but contrary to P.

(13.) In every case where there is constraint, there will be a reaction of constraint resisting the motion of the particle in a particular direction. It is always necessary to remember this in mechanical problems, and to make due allowance for it. The thing to be determined in most cases is the *direction* of the reaction of constraint, and for this purpose the following statements may be useful to the beginner:—

1. When the constraint is produced by the *pull or tension of a string*, the reaction of constraint acts *along the string*.

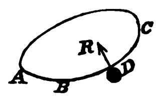
2. When the constraint is produced *by a smooth groove or thin tube*, in which the particle is placed,

the reaction of constraint is *at right angles to the groove or tube.*

3. When the constraint is produced by a smooth hard *plane or surface*, on which the particle is placed, or against which it is pressed by any forces, the reaction of constraint is *at right angles to the plane or surface.*

4. When a smooth body, such as ABC, (fig. 34,) is pressed against an obstacle, such as D, a force of constraint R is exercised by the obstacle upon the body at right angles to the smooth surface ABC of the body.

Fig. 34.



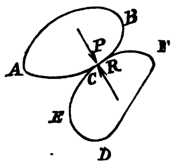
(14.) *Definition of Smoothness.*—The statements just made will be made more evident when we explain precisely what is meant by the word *smooth*. A surface is said to be *perfectly smooth* when it has no power of resisting in the least degree the motion of a body *along it*. Thus, a sheet of ice is nearly a perfect smooth, for the least force will make bodies slide along it. On the contrary, a surface of wood or metal is by no means smooth; if a body be placed on such a surface, it will not slide along it, as in the case of ice, but will require a certain amount of force to push it on. This, then, being the nature of a perfectly smooth surface, that it can exert no force of any kind to resist the motion of a body sliding along it, it is clear that the only kind of force it can exercise is a perpendicular resistance, that is, a resistance acting at right angles to the surface.

Accordingly, *we define a smooth surface* to be one which cannot exercise any resistance or force of constraint, *except at right angles to itself.*

(15.) *Hence, if smooth surfaces*, such as A B C,

DEF, (fig. 35,) be in contact at the point *C*, and be pressed against each other by any forces, and if *P* represent the pressure of *A B C* against *DEF*, and *R* the return pressure or reaction of *DEF* against *ABC*, the directions of *P* and *R* must lie in the *common perpendicular to the surfaces* at the point where they touch, or, as it is called, *the point of contact*. We speak of the *common* perpendicular, because, whatever line is perpendicular to one surface, must also be perpendicular to the other, at the point *C* where the two surfaces touch; otherwise the surfaces would not *touch* but *cut* each other at the point *C*. In fact, when two bodies touch each other, their surfaces must be coincident in the immediate vicinity of the point of contact, and therefore a line at right angles to one surface at the point of contact, must also be at right angles to the other surface.

Fig. 35.



STATICAL AXIOMS.

I.

We may always remove a set of forces, if we put their resultant in place of them.

Also we may remove a force, if we put in place of it a set of forces of which it is the resultant.

(16.) The truth of this axiom is manifest, since the resultant produces the same effect as the forces, (see definition of "*resultant*,") and therefore may be substituted for them at pleasure, or they for the resultant.

This axiom amounts to this,—that we ma

compound forces into their resultant, or *resolve* a force into its components, at pleasure.

II.

Forces which balance each other may be removed or applied at pleasure.

(17.) If forces balance each other, they produce no effect; and therefore, if they be removed from acting on a body, no change can be produced by their removal; or, if they be applied to the body, no alteration can result from their action. In fact, balancing forces virtually destroy each other, and it is therefore immaterial whether they act on a body or not, so far as the motion or rest of that body is concerned.

III.

When a set of *free* and *unconnected* particles have no tendency to approach towards or recede from each other, we may, if we please, suppose them to become *rigidly connected* with each other, without thereby disturbing or producing any motion among them.

(18.) The truth of this axiom is evident if we consider that, since the particles have no tendency to approach towards or recede from each other, it is of no consequence whether we suppose them capable of such motion or not. We may therefore suppose them to be incapable of such motion, or, in other words, we may suppose them to become rigidly connected with each other.

(19.) According to this axiom, we may suppose any flexible or fluid body, which is in a state of equilibrium, to become a rigid body without thereby *disturbing it in* any way. For, if it be

in a state of equilibrium, its particles have no tendency to approach towards or recede from each other; and therefore we may suppose them to be incapable of such motion without disturbing their equilibrium.

IV.

If any point of a *free* body has no tendency to move, we may suppose it to become a *fixed* point if we please.

(20.) For, since the point has no tendency to move, it is of no consequence whether we suppose it to be capable of motion or not. We may therefore suppose it to become incapable of motion, that is, a fixed point, without thereby producing any effect on, or change in, the condition of the body.

V.

Generally, if particles be *capable* of moving in a certain manner, but have *no tendency* to do so, we may, if we please, suppose them to become *incapable* of so moving.

VI.

The resultant of two or more forces acting on a particle *in the same direction*, is equal to the *sum* of them, and acts in the same direction.

(21.) Thus, if the particle be pulled vertically downwards by three forces, one, 4 lbs., another, 6 lbs., and the third, 9 lbs., it is clear that the whole force pulling the particle downwards is $4 + 6 + 9$, or 19 lbs.; that is, the resultant of the forces is the sum 19 lbs. acting in the same direction as the forces.

VII.

The resultant of two forces acting on a particle *in opposite directions*, is equal to their *difference*, and acts the same way as the greater of the two forces.

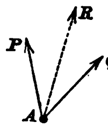
(22.) Thus if the particle be pulled vertically downwards, by a force of 10 lbs., and upwards by a force of 6 lbs., it is clear that there is an unbalanced force of 10 — 6, or 4 lbs. acting downwards; that is, the resultant of the two forces is the difference, 4 lbs., acting downwards, or in the direction of the greater of the two forces.

VIII.

The resultant of two equal forces, acting on a particle at an angle to each other, is a force whose direction bisects the angle made by the two forces.

(23.) Let P and Q, fig. 36, be the two forces acting on the particle A, and let R represent the resultant: then, since P and Q are equal, their joint effect must be to produce a tendency to motion in a direction as much inclined to P as to Q, or, if we may so speak, *half-way* between P and Q. In other words, the resultant R must lie half-way between P and Q, and therefore divide the angle PAQ into two equal parts, or, as it is termed, *bisect* it.

Fig. 36.

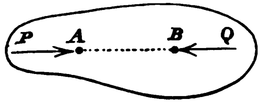


IX.

Two *equal* forces acting in *opposite directions* on a particle, *balance each other.*

X.

Fig. 37.



Two *equal* forces, P and Q, acting at the points A and B, fig. 37, of a rigid body, in exactly *opposite directions*, balance each other.

XI.

Fig. 38.

Two *equal* forces, P and Q, pulling a string AB, fig. 38, in opposite directions, balance each other; the string being supposed to be inextensible.

XII.

In the three foregoing cases (Axioms IX. X. XI.) the forces P and Q will *not balance* each other, except they be *equal* and *opposite*.

(24.) These Axioms are of perpetual use in Statical propositions, and ought to be specially attended to. Others might be added, but we defer them till they are required, in order to avoid too much of this kind of statement at first.

STATICAL PRINCIPLES.

I.

Principle of the transmission of force through a rigid body.

If a force act upon a rigid body, it is *transmitted* by the rigid body *along its line of direction*, and may be supposed to act at any point of that line.

(25.) This principle is often regarded as an *experimental* principle, but it may be easily ex-

plained and proved, without appeal to experiment, as follows:—

Let P, fig. 39, be the force acting at the point A of the rigid body; let the dotted line AB be the direction of P produced, and let C be any point of that line. We may, by Axiom II., apply at C two forces, Q and R, each equal to P, and acting along the line AB, in opposite directions; for these two forces balance each other, by Axiom IX. Suppose this to be done; then we have three forces, P, Q, and R, acting along the line AB on the rigid body. But R and P are equal and act in opposite directions: they therefore balance each other, by Axiom X., and we may consequently remove them if we please, by Axiom II. Now, supposing this to be done, there remains only the force Q, as is shown in fig. 40. Hence it follows that we may remove P, and put Q in its place: in other words, we may suppose P to be transferred, along its line of direction, so as to act at C instead of A. This is what is meant when we say, that P is *transmitted* along its line of direction, and may be supposed to act at any point of that line.

Fig. 39.

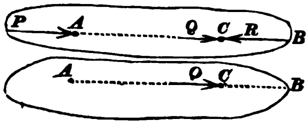
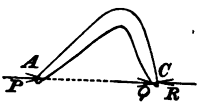


Fig. 40.

(26.) In what has just been said, there is nothing that supposes the rigid body to be of any particular shape: it may, for instance, be of any bent form, as is shown in fig. 41. In other words, the line $A\,C$ may fall outside the body or *not, as the case* may be. All that is

Fig. 41.



necessary is, that the points A and C should be rigidly connected with each other, which is equally the case in figs. 39 and 41.

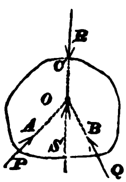
This principle is made use of with great advantage in a number of important problems.

II.

Principle of the concurrence of three balancing forces.

When *three forces*, acting on a rigid body, balance each other, their directions, produced if necessary, must meet at the *same point*, or *not at all*.

Fig. 42.



(27.) Let P, Q, and R, be the three balancing forces, acting at the points A B C, fig. 42, and let the directions of P and Q meet at the point O. Then P and Q have some resultant, which, of course, acts at the point O, for, by Principle I. we may suppose P and Q to act there. Let S represent that resultant, whatever it may be; and let S be put in place of P and Q, which may be done without disturbing the equilibrium, (Axiom I.) S and R, therefore, balance each other, and must, consequently, be equal and opposite forces, (Axiom XII.) Therefore, R lies in the same line as S, and therefore the direction of R, produced if necessary, must go through the point O, and the three directions consequently meet there.

We have here supposed, in starting, that the directions of two of the forces meet; if this be not true, that is, if no two of the directions can be made to meet, it is clear that the three directions *must be* parallel lines; otherwise, two of them

would be sure to meet somewhere. This is the reason that we have introduced the words, "*or not at all*," in the enunciation of the principle. In fact, it amounts to this, that if the three directions be not parallel lines, they must all meet at the same point.

(28.) The shape of the rigid body is of no consequence in this principle; it may be straight, or bent, or round, or flat. Nor is it necessary that the point C, where the forces meet, should be inside the body, or form a part of it. For, suppose a hole to be made in the body all round the point C, or the body to be scooped out in any way so as to cut away the portion which contains C altogether, this will not affect the action of the forces, provided we do not interfere with the points A, B, and C, where the forces really act. All that is really necessary is, that these points be rigidly connected, or, in other words, that the rigidity of the body be neither destroyed nor sensibly weakened, by cutting away the portion containing the point C, or otherwise altering the shape of the body.

Whatever, then, be the shape of the body, the directions of the three forces, produced if necessary, must meet in the same point, or be parallel to each other; otherwise, there cannot be equilibrium.

(29.) We may observe, that, though this condition is *necessary*, it is not *sufficient* for equilibrium; that is, the three forces *must* meet in the same point, if they balance each other: but they *may* meet in the same point, and yet not balance each other; for it is necessary for equilibrium, that S and R shall be equal, as well as act in opposite *directions; which* may not be the case,

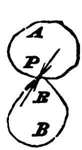
though the condition of the three forces meeting in the same point be satisfied.

III.

Principle of the equality of action and reaction.

Where two bodies are pressed against each other by the action of any forces, the *reaction* of one body is equal and opposite to the *action* of the other body against it.

Fig. 43.



Let A and B, fig. 43, be the two bodies, which are supposed to be pressed against each other by the action of any forces; let P represent the pressure, or *action*, which A exercises against B, and R the return pressure, or *reaction*, which B exercises against A; then P and R must be equal forces, and act in opposite directions.

(30.) This is an *experimental* principle; and all that we can say in proof of it is, that it is found to be invariably true, whatever may be the nature of the bodies, and whatever may be the manner in which the mutual action and reaction between them takes place.* It extends also to the mutual attractions of the heavenly bodies upon each other; for instance, though it may seem strange to persons commencing the study of Mechanics, the force which the sun exercises upon the earth, in virtue of the attraction of gravitation, is exactly equal to the force which the earth exercises upon the sun, though the earth be so small compared with the sun.

(31.) We shall conclude this chapter with some

* This principle is one of Newton's Laws of Motion; it is *necessary to* introduce it in Statics, because it must be *continually* assumed in statical problems.

problems and examples, illustrative of what has been said, and especially of the use of Principle II.

PROBLEMS AND EXAMPLES.

(32.) Ex. 1.—What is the resultant of the forces 4, 7, 9, and 15,* acting vertically upwards on a particle? (See Axiom VI.)

Ex. 2.—What is the resultant of a force 17 acting on a particle towards the right, and a force 10 in the opposite direction? (Axiom VII.)

Ex. 3.—What is the resultant of the forces 7 and 5 acting upwards, and the forces 8, 3, and 12 acting downwards, on a particle?

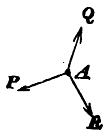
In this example, the upward forces 7 and 5 are equivalent to an upward force 12, and the downward forces to a downward force 23; therefore, by Axiom VII. the resultant of the whole set is $23-12$, or 11 acting downwards.

Ex. 4.—What is the resultant of the forces 6 acting upwards, 4 downwards, 3 downwards, 7 upwards, and 2 downwards?

Ex. 5.—What single force will balance the following forces acting together, namely, 4 acting upwards, 3 acting towards the right, 14 in the opposite direction, 6 downwards, and 2 upwards?

Ex. 6.—Assuming, (what is manifest,) that, if P, Q, and R, fig. 44, be three equal forces acting on a particle A, and making the angles PAQ, QAR, RAP, each equal to 120°, they will balance each other; find the resultant of P, Q, and R, supposing that P is 10, Q 10, and R 16.

Fig. 44.



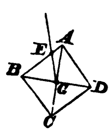
* When we speak of *forces 4, 7, 9,* &c. we mean forces of 4 lbs., *7 lbs., 9 lbs., &c.*

Ex. 7.—In the same case, find the resultant when P is 5, Q 3, and R 5.

Ex. 8.—Find the resultant of two forces, each equal to 5, acting on a particle, and making an angle of 120° with each other.

(33.) In the following examples are cases of bodies suspended or supported at some particular point; in such cases, the centre of gravity (see page 31) must lie vertically beneath the point of suspension or support. That this must be the case is evident from Axiom XII.; for the body is acted on by two forces, namely, the weight of the body acting vertically downwards, and the supporting force acting at the point of support. Now, by Axiom XII. these two forces must act in exactly opposite directions, which cannot be, unless the direction of the former force, produced if necessary, passes through the point of support; that is, the centre of gravity (at which this force acts in a vertical direction) must be vertically beneath the point of support.

Fig. 45.



Ex. 9.—A square $A\,B\,C\,D$, fig. 45, is suspended from the point E, whose distance from A is one-fourth of $A\,B$, find what angle $A\,B$ makes with the vertical.

In this example, the square must be drawn carefully on paper, and a line from E through the centre of gravity G, which is at the intersection of the two diagonals. The line $E\,G$ must be vertical when the square is suspended at the point E; and therefore, if we measure the angle G E B, we find the

angle which $A B$ makes with the vertical, as required.

Ex. 10.—Where must the point of suspension be in the side $A B$, so that $A B$ may make an angle of 30° with the vertical?

Ex. 11.—A triangle, whose sides are respectively 3, 4, and 5 feet long, is suspended by the angular point where the sides 3 and 4 intersect: find at what angle the side 5 is inclined to the horizon.

The centre of gravity of a triangle is at the point of intersection of the lines drawn from any two of the angles to the middle points of the opposite sides.

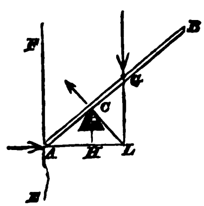
Ex. 12.—A rectangle, whose sides are 4 and 9, is suspended by one of the angles: what angle does the side 4 make with the vertical?

(34.) The following problems and examples are solved by means of the principle of the concurrence of three balancing forces, which is applicable in a great number of cases without the aid of any other statical principles.

Problem I.

$A B$, fig. 46, represents a smooth rod or beam, which rests upon a smooth prop C, and the extremity A rests against a smooth vertical plane or wall $E F$: it is required to determine how long the beam must be in order to make a given angle with the vertical.

Fig. 46.



(35.) Let G be the centre of gravity of the beam (that is, the middle point);

draw AL perpendicular to the wall EF, CL perpendicular to the beam AB, and GL vertically downwards. Then the beam rests upon the prop, and against the wall, and, therefore, in virtue of its weight, it must exercise a certain amount of pressure on the prop and wall: consequently, the prop and wall will exercise corresponding reactions on the beam, (see page 92.) Hence, the forces which act on the beam are, first, its weight acting at G vertically downward, along the line GL; secondly, the resistance, or reaction, of the prop acting at C at right angles to the beam AB, that is, along the line LC produced; and thirdly, the resistance or reaction of the wall at A acting at right angles to the wall, that is, along the line AL. We say that the reactions of the prop and wall are at right angles to the beam and wall respectively, because the beam, prop, and wall are supposed to have smooth surfaces. On this point, see what has been said on the subject of smoothness above, (page 83.)

Now, by Principle II. these three forces must meet in the same point; or, in other words, since L is the point where two of the forces meet, namely, the reaction of the wall and prop, the third force, that is, the weight acting along GL, must pass through L. It follows, therefore, that the line GL, which is drawn vertically downwards from the middle point G of AB, must pass through L, the point of intersection of the line AL (which is perpendicular to the wall), and the line CL, which is perpendicular to the beam.

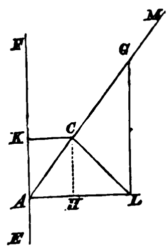
(36.) If, therefore, we draw the line AB, making the proper angle whatever it may be with *the wall*, and then the perpendiculars AL and

$C L$, in which way we find the point L where they meet; all we have to do is to draw $L G$ vertically to meet $A B$ in G, and G must be the middle point of $A B$. Wherefore, if we measure $A G$, we shall know the half, and consequently the whole length of $A B$. We shall now give some numerical examples.

Ex. 1.—If the prop is one foot horizontally from the wall, how long must the beam be, so as to rest at an angle of 30° to the vertical?

Draw a line $E F$, fig. 47, to represent the wall, and $K C$ perpendicular to $E F$, measuring $K C$ equal to 1, (see what has been said respecting the diagonal scale in Part I. page 69;) then draw through C a line $A M$, making the angle $K A M$ equal to 30°, or, what is easier, and comes to the same thing, the angle $K C A$ equal to 60°;* this line represents the direction of the beam. Having done this, draw $A L$ perpendicular to $K A$, and $C L$ perpendicular to $C A$, to meet at L, and draw $L G$ vertically to meet $A M$ at G: then measure the length of $A G$ with the compass and diagonal scale, and twice the length so found will be the length of the beam, as required.

Fig. 47.



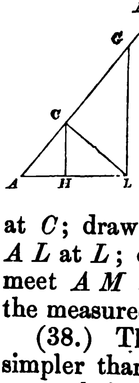
(37.) We shall stop to make some remarks of importance here relative to this and similar constructions, and which, if properly attended to,

* $K C$ represents a horizontal line, and $K A$ a vertical; and if $A B$ makes an angle of 30° with the vertical, it must be inclined at 60° to the horizon.

will often greatly simplify the solution of problems by the graphical method.

Simplification of constructions in the graphical solution of problems.—In the example we have just treated, there are some lines drawn which are not really essential to the determination of the result, though they tend to make the principle on which the solution depends more obvious. Our object is, to find the length of the line $A\,G$ by measurement; and what we have given for this purpose is, that the line $C\,K$ is 1, and the angle $C\,A\,K$ 30°. Now, it is easy to see that the angle $C\,A\,L$ is 60°; and that, if we draw $C\,H$, as in fig. 48, at right angles to $A\,L$, $A\,H$ will be equal to 1.

Fig. 48.



Hence, we may make the construction as follows: draw two lines $A\,M$, $A\,L$, fig. 48, making an angle of 60° with each other, and measure a portion $A\,H$ equal to 1; draw $H\,C$ at right angles to $A\,H$ to meet $A\,M$ at C; draw $C\,L$ at right angles to $A\,C$ to meet $A\,L$ at L; draw $L\,G$ at right angles to $A\,L$ to meet $A\,M$ at G; then measure $A\,G$, and twice the measured length will be the result required.

(38.) This mode of construction is really simpler than that which we gave before; though so much is not gained in the present case, in the way of simplicity, as in many other cases, where the difference between a simplified construction and a non-simplified construction is considerable. The rule for simplifying the construction in any problem is simply this: draw a *rough figure,* to represent all the different lines, angles, &c. requisite, and consider what parts of this figure are

really essential to finding the result, and what are not. Then draw a *correct figure*, leaving out all non-essential parts, and determine the result required by measurement.

In many cases, some little alteration in the way of drawing the figure may simplify the process, as, for example, the drawing the line $C H$, in fig. 48, instead of having to draw the lines $E F$ and $C K$ in fig. 47.

Ex. 2.—If the beam make an angle of 60° with the wall, instead of 30°, find its length.

Ex. 3.—If the angle be 45°, find the length of the beam.

Ex. 4.—If the prop be 6 feet horizontally from the wall, find how long the beam must be, to rest at an angle of 60° to the wall.

Ex. 5.—How far horizontally must the prop be from the wall, when the length of the beam is 10 feet, and its inclination to the vertical 45°?

Fig. 48 will solve this example. Begin by drawing $A M$ and $A H$ at an angle of 45° to each other; then make $A G$ equal to 5, and draw in succession $G L$, $L C$, $C H$.

Ex. 6.—If the beam be 16 feet long, and rest at an angle of 60° to the vertical, find the distance of the prop.

Problem II.

If the wall in the preceding problem be inclined to the vertical at a certain angle, to determine the same things as before.

(39.) In this problem the principle of solution is exactly the same as before, and the same lines are *to be drawn; only* the line $E F$, repre-

senting the wall, is to be drawn at the proper angle to the vertical; and, therefore, the angle which the beam makes with the wall will no longer be the same thing as the angle the beam makes with the vertical. Attending to these points, and taking care to draw $A\,L$ not horizontally, but at right angles to the wall, there will be no difficulty in solving the following examples.

Ex. 1.—The wall makes an angle of 30° with the vertical, the beam inclines on the other side of the vertical at an angle of 30° also, and the prop is 2 feet horizontally from the wall; find the length of the beam.

Fig. 49.

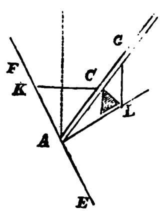


Fig. 49 shows the construction; in this case, the dotted line represents the vertical, and the letters mean the same as before.

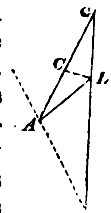
Ex. 2.—The wall makes an angle of 30° with the vertical, the beam inclines on the same side of the vertical at an angle of 60° to the vertical, and the length of the beam is 10 feet; find the distance of the prop from the wall horizontally.

Ex. 3.—The beam is inclined at 20° to the vertical, the prop C touches the beam half-way between A and G; find the inclination of the wall to the vertical.

(40.) The following is the simplified construction in this case, and it affords a good instance of what may be gained by such simplification. Draw *any line* $A\,G$, divided into two equal parts at C,

fig. 50; draw GL, making the angle AGL equal to 20°, to meet CL, which is perpendicular to AG, at L; join L and A, and draw the dotted line at right angles to AL. Then AG represents half the beam, GL the vertical, and the dotted line the wall. The angle which the dotted line makes with GL produced, is the angle required: but it is not necessary to draw the dotted line at all; for, since AL is perpendicular to the wall, and GL is vertical, the angle GLA must exceed by 90° the angle at which the wall is inclined to the vertical; and therefore, if we measure the angle GLA, and deduct 90°, we obtain the result required.

Fig. 50.

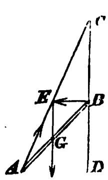


Problem III.

AB, fig. 51, is a beam, of which one end A is held up by a string, AC; and the other end, B, rests against a smooth vertical wall, CD; it is required to determine in what direction the string must pull so as to keep the beam at rest.

(41.) The forces which act on the beam are—its weight acting vertically downwards, at its middle point G; the reaction of the wall at right angles to the wall, that is, horizontally; and the tension of the string along the string. Hence, by Principle II., if we draw BE horizontally, and GE vertically to meet at E, the string must pass through the point E. This determines in what direction the string must pull, as required.

Fig. 51.



Ex. 1.—*What angle* must the string make with

the vertical, so that the beam may rest at an angle of 45° to the vertical.

The simplified construction in this case only requires the portion AEB of the figure to be drawn.

Ex. 2.—The angle ACD is 30°, AC is 10 feet; find how long AB must be so as to remain at rest.

In this example it is important to notice that E is always the middle point of AC, because G is the middle point of AB, and the lines GE and BC are parallel.

Ex. 3.—If the wall be inclined 30° to the right of the vertical, and the beam 45° to the vertical, as in Ex. 1; find the angle which the string must make with the vertical.

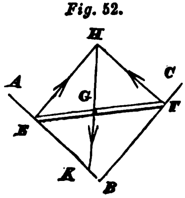
In this example BE is not horizontal.

Ex. 4.—The string is inclined at an angle of 30° to the vertical, and the beam at an angle 60°; find what must be the inclination of the wall in order that the beam may rest.

Problem IV.

AB and BC are two smooth planes inclined to the horizon, and EF is a beam resting with its extremities on the planes; having given the inclination of one of the planes, and of the beam, to find what the inclination of the other plane must be to keep the beam at rest.

Fig. 52.



(42.) The forces which act on the beam are—its weight, acting vertically downwards at its middle point G; the reaction of the plane BA, acting at E at right angles to BA; and the reaction of the plane

BC acting at F, at right angles to BC. Hence, by Principle II., if we draw a vertical line, GH, through G, a line EH perpendicular to BA through E, and a line FH, perpendicular to BC, through F; these three lines must meet at the same point, H, as we have represented in the figure, otherwise the beam will not remain at rest.

Now suppose that the inclination to the vertical of the plane BA, and that of the beam are given, and it is required to find the inclination of the other plane BC; we may easily do so as follows: draw a line HG to represent the vertical, and through any point G of it draw a line EF, making GE equal to GF, and the angle HGF equal to the given angle of inclination of the beam to the vertical; through E draw the line AB, making an angle with HG equal to the given angle of inclination of the plane to the vertical; through E draw EH at right angles to AB, to meet GH at H; join H and F, and draw CB through F at right angles to HF; then, by measuring what angle CB makes with HG, we determine the required inclination of the plane BC to the vertical.

(43.) *Simplified construction.*—This construction may be greatly simplified; for in fact we need only draw the lines EF, HG, HE, and HF in the following manner: draw a line HG to represent the vertical, and EF through G, making the angle HGF equal to the given angle of inclination of the beam; draw from H the line HE, making the angle GHE equal to the *complement**

* The *complement* of an angle is the number of degrees which must be added to that angle to make up, or *complete*, 90 degrees, or a right angle. Thus, 30° is the complement of 60°, because $30° + 60° = 90°$.

of the given angle of inclination of the plane *A B* to the vertical, and let this line meet *G E*, produced if necessary, at *E*; make *G F*, produced if necessary, equal to *E G*, and join *H F*; then measure the angle *G H F*: the complement of this angle will be the required angle of inclination of the plane *B C* to the vertical.

(44.) All that we assume in thus simplifying the construction is, that if a line, such as *A B*, makes a certain angle with the vertical, the line *E H*, which is perpendicular to *A B*, makes the *complement* of that angle with the vertical. This follows from the 32d Proposition of the 1st Book of Euclid; for, by that proposition, the three angles of the triangle *E H K* make together two right angles, or 180°; but the angle at *E* is 90°; therefore, the angles at *H* and *K* make together 90°; or, in other words, either one of these angles is the complement of the other. If the student is not acquainted even so far with Euclid, he may convince himself of the truth of what is stated by drawing a proper figure of *E H K*, and measuring the angles at *H* and *K*.

Ex. 1.—The inclination of *A B* to the vertical is 70°; What must be the inclination of *B C*, so that the beam may rest at an angle of 40° to the vertical?

Ex. 2.—The inclination of *A B* to the vertical is 60°; What must that of *B C* be, so that the beam may rest at an inclination of 45° to the vertical?

Ex. 3.—Show that it is not possible for *A B* to be inclined at angle 60° to the vertical, and the beam at an angle of 30°.

Ex. 4.—Show generally that the inclination of

the beam to the vertical cannot be equal to the inclination of either of the planes to the horizon.

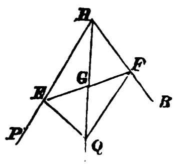
PROBLEM V.

Having given the inclinations of the two planes, in the preceding problems, to the vertical, it is required to find the inclination of the beam.

(45.) In this case the angles GHE and GHF, being the complements of the given inclinations of the two planes to the vertical, are known; we may therefore draw the lines HE and HF, but we do not know where the points E and F ought to be. Now it is clear that all we have to do, to determine these points, is, to draw the line EF so that it shall be divided into two equal parts at G; and this may be done in the following manner:

In fig. 53, draw three lines HP, HQ, HR, from the point H, making the angles PHQ and RHQ equal to the complements of the given inclinations of the planes to the vertical; take any point Q in HQ, draw QE and QF parallel respectively to HR and HP, and join EF; then EF is so drawn that the point G where it meets HQ is half way between EE and F.* The line EF thus drawn represents, therefore, the inclination of the beam to the vertical HG, and we have only to measure

Fig. 53.



* For $EHFQ$ is by construction a parallelogram (or figure bounded by two pairs of parallel lines), and it is obvious that the diagonals (the lines joining the opposite corners) of such a figure bisect each other, as may be proved by measurement, if the student has *not read Euclid, Prop.* 34, Book I.

the angle HGF, which is the required angle of inclination.

(46.) It may be objected to this mode of construction, that we have not attended properly to the length of the line EF, so as to make it represent the length of the beam; but the length of the beam is not, and need not be given, for the result will be always the same, whether the beam be long or short, as it is easy to see; besides, in the figure the length of the beam is represented, not actually, but in miniature, as it were, with reference to a certain scale of small equal parts; and, as we may choose any scale we please (whether the units be inches, or half inches, or quarter inches, or otherwise), the line EF may be of any length we please, and still correctly represent the length of the beam.

Ex. 1.—The inclinations of the planes to the vertical are respectively 30° and 60°; find at what inclination the beam will rest.

Ex. 2.—The inclinations being 45° and 75°, find that of the beam.

Ex. 3.—If the planes be inclined at an angle of 90° to each other; show that the inclination of the beam to the vertical is always double the inclination of one of the planes to the horizon.

In this example, the student who does not know Euclid, Book I., must prove what is stated by showing it to be true in one or two particular cases. By the help of Euclid, Book I., it may be very easily proved to be true in all cases. Observe: the inclination of the plane to the horizon is the complement of its inclination to the vertical.

Ex. 4.—If BC be a vertical plane, and AB

inclined at 45° to the vertical, find at what inclination the beam will rest.

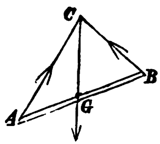
Ex. 5.—The two planes are inclined at an angle of 60° to each other, and the beam rests at an inclination of 30° to the vertical; find at what angle each plane is inclined to the vertical.

Observe, here the angle $E H F$ must be 120°. This is rather a difficult example, and requires a knowledge of the 3d Book of Euclid.

Problem VI.

A beam $A B$, fig. 54, of given length, is held up by two strings $A C$ and $B C$, also of given length, fastened to a point C: find at what inclination to the vertical the beam rests.

Fig. 54.



(47.) If G be the middle point of the beam, the vertical line through G must pass through the point C, for C is the point where the two forces or tensions exercised by the strings on the beam meet, and therefore the direction of the third force, namely, the weight of the beam acting vertically at G, must also pass through C.

Hence, if we construct a triangle $A B C$, having its sides of the proper lengths, as given in the problem,* and if we draw a line from C to the

* The way to do this is as follows: draw one side, say $A B$, of the proper given length; with A as centre, describe a circle, having its radius equal to the given length of $A C$; with B as centre, describe another circle, having its radius equal to the given length of $B C$; let C be one of the points where the two circles meet; then, joining C with A and B, $A B C$ so formed will be *the required triangle.*

middle point G of $A B$, $C G$ will represent the vertical, and $C G B$ the angle of inclination to the vertical at which the beam rests.

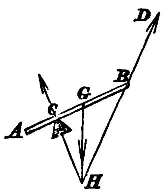
Ex. 1.—A beam, whose length is five feet, is suspended by its extremities from a point by two strings, one 3 feet long, the other 4 feet: find the inclination of the beam to the vertical.

Ex. 2.—The strings are, one 4 feet, and the other 8 feet long, and the beam is 10 feet: find the inclination of the beam.

Ex. 3.—The beam hangs at an inclination of 45°, and the strings are, one 3 feet, and the other 5 feet long: find the length of the beam.

Problem VII.

Fig 55.



A smooth beam, A B, fig. 55, rests on a smooth prop C, one end B being held up by a string B D: it is required to state what is necessary for equilibrium in this case.

(48.) Draw $C H$ perpendicular to $A B$, and $G H$ vertically through the middle point of $A B$, to meet $C H$ at H; then the pressure of the prop acts along the line $C H$, (see Art. 35,) and the weight of the beam along $G H$. The directions of these two forces, therefore, meet at H; hence, the direction of the third force—the pull or tension of the string—must also pass through H; and therefore, if we join H and B, the string must pull along the line $H B$ produced.

Ex. 1.—C is half way between A and G, and the beam rests at an inclination of 45° to the

vertical; find the inclination of the string to the vertical.

Ex. 2.—If $A\,C$ is double $C\,G$, and the beam's inclination to the vertical is 30°, find that of the string.

Ex. 3.—The beam is 10 feet long, and its inclination to the vertical, and that of the string, are 30° and 60° respectively; find the length of $G\,C$.

Ex. 4.—C is half way between A and G, and the angle $A\,B\,D$ is 150°; What is the inclination of the beam to the vertical?

MATHEMATICAL SOLUTION OF THE PRECEDING PROBLEMS.

(49.) For the sake of those who have advanced in Mathematics as far as trigonometry, we shall briefly show how the problems we have just given may be solved by the aid of mathematical formulæ; from which the examples in each case may be easily deduced, if the student has trigonometrical tables, and knows how to use them.

Problem I. Let $K\,C$, fig. 47, be denoted by a, and the angle $K\,A\,C$ by θ; then

$$\angle\, CLA = 90° - \angle\, CAH, \text{ and } \theta = 90° - \angle\, CAH;$$

$$\therefore\ \angle\, CLA = \theta.$$

Also,

$$\angle\, CGL = 90° - \angle\, CLG, \text{and} \angle\, CLA = 90° - \angle\, CLG;$$

$$\therefore\ \angle\, CGL = \theta.$$

Hence, if we denote AG by b, we find

$$KC = A\,C \sin.\theta,\ A\,C = AL \sin.\ \theta,\ A\,L = AG \sin.\theta\,;$$

CHAPTER II.

OF THE PRINCIPLE OF THE LEVER.

(51.) AFTER the preliminary statements and explanations given in the preceding chapter, the natural course to pursue would be, in the first instance, to prove and explain the various rules for the composition and resolution of forces, so as to be able to find the resultant of any two or more forces, or the components of which any force is the resultant. It will be advisable, however, to defer this to the next chapter, and to devote the present chapter to the consideration of the mechanical power or instrument called the *lever*; for the principle upon which questions relating to the lever are solved, admits of a very simple and intelligible demonstration; and a variety of problems and examples deducible therefrom may be easily solved, and with considerable advantage to a beginner. Besides, the fundamental proposition relating to the composition and resolution of forces, called the *Parallelogram of Forces,* which we shall give in the next chapter, is deducible from the principle of the lever. For these reasons, and in order to make the student's progress more easy, *it will* be advisable to introduce the consideration

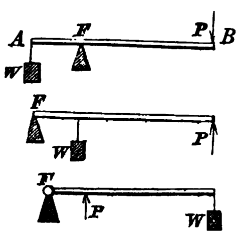
of the lever here, though it is not exactly in its proper place.

(52.) *Definition of a Lever.*—A rigid body in which there is a fixed point or axis, round which it may freely turn, is called a *lever*.

Fulcrum.—The fixed point or axis about which the lever may turn, is called its *fulcrum*.

(53.) *Different kinds of Lever.*—The name lever is derived from the Latin word, signifying *to raise* or *elevate*; it is generally applied to any strong bar, such as a crow-bar, used for the purpose of raising great weights, or displacing obstacles. There are three kinds of lever, distinguished from each other by the position of the fulcrum with reference to the power employed to move the lever, and the resistance to be overcome by it. Fig. 56 shows a *lever of the first kind*; *AB* is the bar, and *F* the fulcrum; *W* is the weight to be raised or resistance to be overcome, and *P* is the power employed to move the lever. In this kind of lever the fulcrum is always between the power *P*, and the weight or resistance *W*. Fig. 57 shows a *lever of the second kind*, in which the power *P* and resistance *W* are on the same side of the fulcrum, and the power is further from the fulcrum than the resistance. Fig. 58 shows a *lever of the third kind*, in which, as in those of the second kind, the power *P* and resistance *W* are on the same side of the fulcrum, but the power is nearer to the fulcrum than the resistance.

Figs. 56, 57, 58.



(54.) In levers of the first and second kind, *power is gained and speed is lost;* that is, the power (P) necessary to move the weight (W) is smaller than the weight, but the speed with which the weight is moved is proportionally less than that at which it is necessary that the power should move. For example, if P be 10 times further from the fulcrum than W, and if W be 10 lbs., P need only be barely over 1 lb. to move W, but, at the same time, P must go over a space of 10 inches to make W move an inch.

In the case of levers of the third kind, the reverse is the case, *power is lost and speed is gained;* that is, the power necessary to move the weight is greater than the weight, but the speed of the latter is proportionally greater than that of the former.

(55.) It is an important law in Mechanics, that *whatever is lost in power is gained in speed,* and *whatever is gained in power is lost in speed.* The levers of different kinds afford a good example of this, as has just been explained; but the law is not restricted to levers, it applies equally to every kind of machine or contrivance, as may be proved by what is termed the *Principle of Work,* to which we shall call the student's attention hereafter. From ignorance of this principle, or from inattention to it, many persons have wasted their time and money in trying to invent machines for gaining power without losing speed; for all the contrivances proposed for producing a *perpetual motion* resolve themselves into this error, or something nearly akin to it.

(56.) *Examples of the different kinds of Lever.*—A crow-bar employed to raise a stone is a good

example of a lever of the first kind; the stone to be lifted is the resistance, the man exerts the power by his hand, and the fulcrum is the stone on which the bar is rested, or the *purchase*, as workmen call it, about which the lever is turned. A spade used for digging the ground is also a lever of the first kind; in fact, it is the same thing as a crow-bar, but used for a different purpose, the resistance to be overcome being however much smaller. A pair of scissors is a double lever of the first kind, the joint being the common fulcrum of the two levers, and the material to be cut, the resistance. Pincers, and other similar instruments, the steel-yard used for weighing by butchers, and a variety of implements, are levers of the first kind.

A crow-bar used in the manner represented in fig. 57, becomes a lever of the second kind; here the extremity is rested on the ground, and so becomes the fulcrum, or purchase, about which the bar is turned. The resistance W, and the power P, are both on the same side of the fulcrum, the latter being further from it than the former. A wheelbarrow is a good example of this kind of lever; the centre of the wheel is the fulcrum, and the weight acts at the centre of gravity of the load. A nut-cracker is a double lever of the second kind, the joint being the common fulcrum of the two levers, and the nut to be cracked, the resistance. The oar of a boat is a lever of the second kind, the water being the fulcrum or purchase against which the blade presses.

In all these instances of the use of levers, the object is to gain power, and not speed; where speed *is to be gained, the* lever of the third kind

must be employed. The limbs of animals afford many instances of this kind of lever; thus, the fore-arm of a man is turned about the elbow-joint, by the contraction of a muscle which exerts an upward pull at a point near the elbow-joint. Here the joint is the fulcrum, the power acts near the fulcrum, and the weight on the hand, which is much further off. The muscles are capable of exerting considerable force, and therefore they are made to act at a disadvantage as regards power, in order to gain speed, and communicate rapid motion to the limbs. The chief reason, however, why levers of the third kind are employed in the limbs of animals, is to give compactness of form to their bodies, and make the limbs project from them in such a way as to be capable of moving with convenience. The treadle, or foot-board, used by a knife-grinder, is a lever of the third kind; the foot exerts the power, the string the resistance. A pair of tongs is a double lever of the third kind, and is analogous to the pair of scissors, which exemplifies the first kind, and the nut-crackers the second kind.

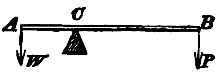
(57.) It is easy to see from these examples, that the lever is an instrument, or *mechanical power*, of which there are many familiar uses: it is also of constant application in machines. There is a very simple rule for calculating what power will balance a given weight by means of a lever, and this rule is applicable to other mechanical powers besides the lever; indeed, there are few mechanical problems or investigations in which it may not be employed. It is commonly called the *principle of the lever*, and it is on account of its general utility in Mechanics, and

its simplicity both as regards proof and application, that we have selected the lever, of all the mechanical powers, and given it a place at the commencement of the treatise. We shall now prove this principle by a very ingenious method, and one of great interest, inasmuch as it was given by Archimedes, the great mechanician, and employed by him as the foundation of all his mechanical demonstrations.

Principle of the Lever.—Proposition I.

If AB be a straight lever, C the fulcrum, and W and P two forces acting at A and B, at right angles to AB, W and P will balance each other if they be proportional to the lines BC and AC respectively, or, as it is said, if they be inversely proportional *to the* arms *at which they act.*

Fig. 59.



(58.) To prove this we must first suppose the following case, viz.: Let *DE*, fig. 60, be a straight rod,* *C* its middle point, *F* any other point, *A* the middle point of *DF*, and *B* the middle point of *EF*.

Fig. 60.

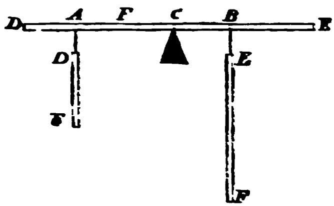
Now, the weight of the whole rod is, or may be supposed to be, exerted at its middle point (or

* We shall always, when we speak of a rod or beam, suppose it to be of uniform weight and thickness, unless the contrary be expressed, and we shall therefore assume that its centre of gravity is at its middle point.

centre of gravity) C, and if we put a support or fulcrum at C, the rod will rest horizontally upon it, without any tendency to turn one way or the other. If, however, we consider separately the effects which the portions DF and EF of the rod produce by their weights, we may suppose the weight of DF to act at its middle point A, and the weight EF at its middle point B.

Hence it follows, that a weight equal to that of DF, acting on the lever at A, and a weight equal to that of EF acting at B, as is represented in fig. 61, produce the same effect as the weight of the whole rod acting as it really does; that is, the two weights suspended from A and B, have no tendency to turn the rod one way or the other, and therefore they balance each other.

Fig. 61.



Now let AD (fig. 60) be denoted by a, and BE by b; then, since A is the middle point of DF, and B that of EF, it follows that

$$DF = 2a \quad , \quad EF = 2b,$$

and therefore

$$DE = DF + EF = 2a + 2b,$$

whence

$$\text{half of } DE = a + b.$$

But, since C is the middle point of DE, DC is half of DE, and $DC = DA + AC = a + AC$: Hence

$$a + AC = a + b,$$

and therefore, taking away a from both, we find,

$$AC = b.$$

In like manner we have

$$EC = \text{half of } DE = a + b;$$

but

$$EC = BC + BE = BC + b;$$

therefore

$$BC + b = a + b:$$

hence, taking away b from both, we find

$$BC = a.$$

Now the weight DF acting at A (fig. 61) is the weight of a length $2a$ of the rod, and the weight EF acting at B is the weight of a length $2b$ of the rod; and, since the rod is uniformly thick and weighty throughout, these weights are respectively proportional to $2a$ and $2b$, or, what is the same thing, to a and b. But we have proved that $AC = a$, and $BC = b$; hence it appears, that the weights acting at A and B are proportional to BC and AC. If therefore W and P, in fig. 59, be proportional to BC and AC, they will balance each other.

Now, AC and BC are called the *arms* at which the forces W and P act, the word *arm* meaning *distance from fulcrum*; hence, when the proportion of the two forces to each other is the *inverse* of that of the arms at which they act, they balance each other; that is, when the force at A is to the force at B in the same proportion, *not* as AC is to BC, *but* as BC is to AC, (for this is the meaning of the word *inverse*,) the forces balance each other. *Which was to be proved.*

(59.) A numerical example may be useful to the student who is not accustomed to demonstrations of this kind, and we shall therefore give the

following: Suppose DE to be 20 feet long, and that each foot weighs 1 lb.; also, let DF be 6 feet, and therefore EF 14 feet; then, since A is the middle point of DF, B that of EF, and C that of DE, it follows that DC is 10, AD is 3, and AC, which is found by subtracting AD from DC, is therefore 7. In like manner, BC is found by subtracting EB, the half of EF, from EC, the half of DE, that is, the half of 14 from the half of 20; therefore, BC is 3.

Now, the rod will rest horizontally upon a prop or fulcrum at C; also, the weight of the portion DF, 6 lbs., may be supposed to act at its middle point A, and the weight of the portion EF, 14 lbs., may be supposed to act at its middle point B. Whence we have the case of a lever, C being the fulcrum, AC being 7 feet, and BC 3 feet, the weight acting at A being 6 lbs., and that at B 14 lbs.; these weights balance each other, and (since 14 is to 6 as 7 is to 3) they are in the same proportion to each other, not as AC is to BC, but as BC is to AC, that is, inversely as AC to BC.

(60.) *Corollary. Pressure on Fulcrum, how found.* —It is evident that the whole weight of the rod DE rests upon the fulcrum C, and therefore the pressure on the fulcrum produced by the weight of the rod, or, what is the same thing, by the weights of the two portions DF and CF, is equal to the weight of the rod: that is, the pressure on C is the sum of the weights of DF and CF.

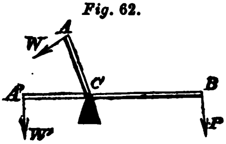
Hence we conclude, referring to fig. 59, that the pressure produced on the fulcrum by the two *forces* P and W, acting on the lever AB, is the *sum of* P and W.

Proposition II.

If the two forces, W and P, act upon a bent *lever, $A\,C\,B$, fig.* 62, *and at right angles to the arms, $C\,A$ and $C\,B$, C being the fulcrum, the forces will balance each other when they are* inversely proportional *to the arms.*

(61.) By a *bent* lever we mean a lever in which the arms are inclined at an angle to each other. To prove this proposition, suppose the line $B\,C$ to be produced to A' so far that $C A'$ shall be equal to $C A$; and imagine $C A'$ to be a third arm of the lever, so that $A\,C$, $A'\,C$, and $B\,C$ form one rigid body; then, if there were a force W' equal to W acting at A', at right angles to $C A'$, as represented in this figure, it is easy to perceive that W' would produce the same effect as W, in tending to turn the lever round the fulcrum C; for, as far as rotation about C is concerned, and with reference to C, the two forces, W and W', are situated in precisely the same manner, and would exert the same power on the lever.

Fig. 62.



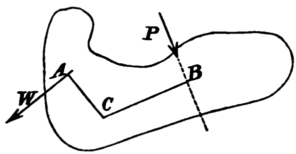
Hence, since W' would produce the same effect in turning the lever as W, we may conceive W' to act in place of W; and then we have a *straight* lever with two forces, W' and P, acting upon it, as in Proposition I. Hence, if the forces balance, W' is to P inversely as $A'\,C$ to $B\,C$; or, since W is equal to W', and $A\,C$ equal to $A'C$, W is to P inversely as $A\,C$ to $B\,C$. The forces, therefore, if they balance each other on a bent lever, acting at right angles to the arms, must be inversely proportional *to the arms. Which was to be proved.*

PROPOSITION III.

If two forces, W and P, act on a lever of any shape, *as in fig.* 63, *and if perpendiculars, CA, CB, be drawn from the fulcrum C, upon the directions of W and P, produced if necessary; the two forces will balance each other when they are* inversely proportional *to the perpendiculars so drawn.*

(62.) For, by the principle of the transmission of force through a rigid body, we may suppose the forces to act at any points of their respective lines of direction, and therefore we may conceive them to act at A and B, as is represented in the figure.

Fig. 63.



Thus we may call ACB a bent lever, and therefore, by the preceding proposition, the forces will balance each other if they be inversely proportional to the two perpendiculars, AC and BC. *Which was to be proved.*

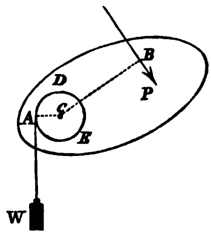
PROPOSITION IV.

To estimate the effect of a force acting on a lever, we must multiply the force by the perpendicular upon it from the fulcrum, that is, we must multiply the number *of pounds in the force by the* number *of feet in the perpendicular.*

(63.) This is easily deduced from the previous proposition, as follows: Let BC, in fig. 64, be the perpendicular let fall from C upon the force P, acting on a lever of any shape, C being the fulcrum; let W be the weight which, acting at an *arm* unity, balances P, that is, let the perpen-

dicular CA upon W be always 1; which will be the case if we suppose W to be a weight hanging from a wheel ADE, attached to the lever, and having its centre at the fulcrum C, as shown in the figure. Then, by the previous proposition, we know that W is to P in the same proportion as CB to CA, or, as it is usually and conveniently expressed by means of dots, observing that CA is 1,

Fig. 64.



$$W : P :: CB : 1.$$

Therefore, by the *Rule of Three*, or, as it is often called, the *Rule of Proportion*, we have the following result,

$$W = P \times CB.$$

Hence it appears, that, if we multiply a force P by the perpendicular AC, let fall upon it from the fulcrum, the result expresses the amount of weight, acting at an arm 1, which P is capable of balancing.

Thus, suppose that P is 10 lbs. and CB 4 feet; then W is 40 lbs.; that is, P can balance 40 lbs. acting at an arm 1. Again, suppose that P is 15 lbs. and CB 6 feet; then W is 80 lbs.; that is, P can balance 80 lbs. acting at an arm 1. In the latter case, therefore, P has double the effect it has in the former. Thus we may see that W is a proper measure of the effect of P, because W always acts at the same arm, whereas P may not; for, in estimating the effect of a force on a lever, we must consider two things, namely, first, the magnitude *of the force*, and secondly, the arm,

or perpendicular distance from the fulcrum, at which it acts. If the magnitude of the force be increased, the effect on the lever is of course also increased in proportion; and if the arm be increased, the same may be said. If, however, the arm be always one given invariable length, the effect of the force will depend simply upon the magnitude of the force, and we need not then think of the arm. Now this is the case with *W*, which always acts at the same arm 1; and therefore the effect of *W* will be proportional to the magnitude of *W* simply, without reference to its arm.

Hence, since *P* balances *W*, the effect of *P* is equal and opposite to that of *W*, and therefore the effect of *P* is proportional to the magnitude of *W*; in other words, the number of pounds in *W* represents, we may say, the energy or power with which *P* tends to turn the lever, and therefore the effect of *P* is measured and estimated by that number.

(64.) We might have supposed that *W* acted at any other known and invariable arm, but we naturally assume 1 foot as the simplest arm we can fix upon.

(65.) This is a very important proposition, and it may be considered as embodying the principle of the lever in its most general form. We see by it, that the effect of a force, *P*, on a lever, that is, the energy or power with which it tends to turn the lever about its fulcrum, varies with the magnitude of the force and the arm at which it acts, jointly, and is estimated by the product of the force into the arm; because that product shows *the amount* of the force which, acting at the

invariable arm unity, has the same energy or power on the lever as P.

(66.) *Moment.*—The product of a force (acting on a lever) into its arm or perpendicular distance from the fulcrum is usually called the *Moment* of the force. Hence the *moment* of a force acting on a lever expresses the energy or power with which the force tends to turn the lever about its fulcrum.

(67.) *Like and Unlike Moments.*—If there be two forces acting on a lever, and tending to turn it opposite ways round the fulcrum, they are said to have *unlike* moments; but, if they tend to turn it both the same way, they are said to have *like* moments. Forces, therefore, whose moments are *unlike, resist* each other; and those whose moments are *like, assist* each other.

Proposition V.

To estimate the effect of several forces acting on a lever, we must find their moments, add the moments tending to turn the lever one way into one sum, and the moments tending the opposite way into another sum; then the difference between the two sums will express the total effect of the forces.

(68.) This is immediately evident, since the moments are the weights, which, acting at the arm 1, produce the same effect as the forces; and these weights, since they act at the *same* arm, may be all put together by addition or subtraction, as the case may be, so as to form a single weight.

For *example, let the* forces be as follows:

10 lbs. at an arm 3	}	tending to turn the lever one way, as the hands of a watch, suppose.*
7 lbs. „ 4		
8 lbs. at an arm 7	}	tending the opposite way.
2 lbs. „ 5		

Then the moments of the first pair of forces are 30 and 28; that is, these forces are equivalent to the weights 30 and 28, each acting at an arm unity, and tending to turn the lever the way the hands of a watch go round; and the weights 30 and 28 thus acting, are equivalent to a weight 58. In like manner the moments of the second pair of forces are 56 and 10, or 66 altogether, tending the opposite way. Now the forces 58 and 66, both acting at an arm 1, but tending to turn the lever opposite ways, are together equivalent to the difference between them, which is 8, tending the same way as the greater force. Hence the effect of the whole set of forces is expressed by the number 8; and they tend to turn the lever contrary to the hands of a watch.

(69.) The first and last of the propositions here proved have been each called the *Principle of the Lever*. Proposition I. is the principle in its simplest form; and Proposition V. in its most general form.

(70.) In speaking of rotation, we shall often employ the method just adopted of specifying the two different kinds of rotation, by reference to the familiar case of the hands of a watch. For brevity, we shall say, when the body turns round

* This is a convenient way of speaking of rotation, in order *to distinguish* between the two different ways in which a body *may turn.*

the same way as the hands of a watch, it is a *forward rotation;* but when the opposite way, it is a *backward rotation.* Thus, to a person looking towards the north, the sun, moon, and stars have a *backward* rotation round the pole; but, to a person looking towards the south, they have a *forward* rotation: in one case the heavens turn about the pole *backwards,* in the other case, *forwards.*

PROBLEMS AND EXAMPLES.

Ex. 1. If W and P be two forces which balance each other, acting on a lever AB, at right angles to AB, C being the fulcrum (see fig. 59), and if $W = 10P$; what part of AB is AC?

Ex. 2. If $AC = 10$, $BC = 3$, and $W = 40$; find P.

Ex. 3. If $AB = 20$, and $W = 4P$; find AC.

Ex. 4. If the pressure on the fulcrum be 50, and $BC = 4AC$; find P and W.

Ex. 5. Supposing that P does not act at right angles to AB, but is inclined to AB at an angle which we shall denote by a; find P, when $W = 100$, $AC = 1$, $BC = 4$, $a = 30°$.

To do this we must draw a proper figure, according to the numbers here given, and measure the perpendicular from C upon the direction of P produced, if necessary. We shall find that the perpendicular is 2; and then we have, by Prop. III.,

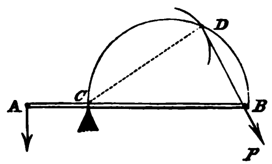
$$2 : AC \text{ (or 1)} :: W \text{ (or 100)} : P;$$

and therefore, $P = \dfrac{100 \times 1}{2} = 50.$

Ex. 6. If $W = 100$, $AC = 2$, $BC = 4$, and $a = 60°$; find P.

Ex. 7. If $P = 20$, $AC = 10$, $BC = 10$, and $a = 45^\circ$; find W.

Fig. 65.



Ex. 8. If $W = 100$, $AC = 1$, $BC = 4$, and $P = 75$; find a.

In this example, let CD, fig. 65, be the perpendicular let fall from C upon DB, the direction of P; then, by Prop. III.,

$$P \text{(or 75)} : W \text{(or 100)} :: AC \text{(or 1)} : CD;$$

$$\therefore\ CD = \frac{100 \times 1}{75} = \frac{4}{3}.$$

Hence, in the right-angled triangle CDB, we know that CB is 4, and $CD\ \frac{4}{3}$, and we can thence find the angle CBD, or a; for, measure a line CB, equal to 4, and on it describe a semicircle CDB;* then with C as centre, and $\frac{4}{3}$ as radius, describe a circular arc cutting the semicircle at D, and draw the lines CD and DB; then DBC is the angle a, and we find a by measuring the angle DBC.

It is important to remember this method of construction, as it is often useful. It depends upon the principle (proved in Euclid, Book III.) that the *angle in a semicircle* is always a *right angle*, that is, that the angle BDC is always a right angle, whatever point of the semicircular circumference D may be. Bearing this principle in mind, it is evident, that, by this method of con-

* By finding the middle point of CB, and describing a half-circle with that point as centre, and half of CB as radius.

struction, we have formed a triangle BDC, in which one angle, D, is a right angle; the opposite side, BC, is 4; and the side DC, $\frac{4}{3}$. This is exactly the triangle required to be constructed, and therefore DBC is the angle required to be found.

Ex. 9. If $W = 50$, $AC = 2$, $BC = 7$, and $P = 60$; find a.

Ex. 10. If P makes an angle 45° with AB, and W an angle 30°, and if $W = 10\,P$; find what part AC is of AB.

Ex. 11. If P and W make equal angles with AB, and $P = 10\,W$; find what part AC is of AB.

Ex. 12. If $P = 10$, and is inclined to AB at an angle 60°, and if $CB = 4$; find the *moment* of P.

Ex. 13. On the same supposition, except that CB is not known, find CB when the moment of P is 400.

Ex. 14. There are three forces, P, Q, and R, tending to turn a lever the same way, and two forces, S and T, tending to turn the lever the opposite way; also, the perpendiculars from the fulcrum upon these forces are respectively p, q, r, s, and t. Supposing that $P = 3$, $Q = 4$, $R = 6$, $S = 9$, $T = 4$, $p = 1$, $q = 3$, $r = 2$, $s = 1$, $t = 10$, find the moment of each of the forces.

Ex. 15. On the same supposition, find the total moment of all the forces.

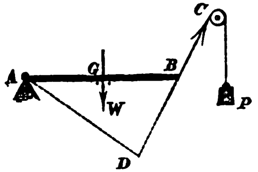
Ex. 16. Supposing P, Q, and S, alone to act on the lever, and that they balance each other, P being 4, $Q = 10$, $p = 1$, $q = 2$, $s = 7$; find S.

Ex. 17. On the same supposition, if $P = 4$, $Q = 10$, $S = 6$, $p = 2$, $s = 3$; find q.

Problem VIII.

If AB, fig. 66, *be a rod or beam, in a horizontal position, having at one end A, a hinge or fulcrum, and the other end B supported by a weight P, hanging by a string which passes over a pulley C, and is fastened to the beam at B; it is required to determine the condition of equilibrium.*

Fig. 66.



Let a perpendicular, AD, be drawn from the fulcrum A to BD, the direction of the string produced; then, the forces which act on the beam are, first, its weight W vertically at its middle point G, and secondly, the tension of the string in the direction BC, which tension is equal to the weight P. Also, the perpendicular distances of these forces from the fulcrum are AG and AD. Hence, by Prop. III. we have,

$$P : W :: AG : AD,$$

which is the condition of equilibrium required.

Ex. 1. If $W = 20$, and the angle $ABD = 30°$; find P.

In this example draw a proper figure, according to the numbers given, and so find AD by measurement; then the proportion obtained in the Problem will give P by the rule of proportion. The length of AB is not given, because it is not necessary that it should be, but it may be assumed of any convenient length, say 10, in which case AG will be 5.

Ex. 2. If $W = 10$, and the angle $ABD = 60°$; find P.

Ex. 3. If $P = 10$, and the angle $ABD = 45°$; find W.

Ex. 4. If $W = 10$, and $P = 8$, find the angle ABD.

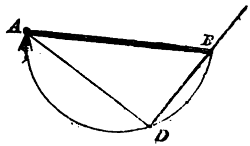
Here, by the proportion, assuming $AB = 10$, we find

$$8 : 10 :: AG : 10;$$

and therefore $AG = 8$.

Hence we must, as in a former case, measure a line AB, fig. 67, equal to 10, on it describe a semicircle; also, with A as centre and 8 as radius, describe a circular arc cutting the semicircular circumference at D: then,

Fig. 67.



drawing the lines AD, BD, the triangle ABD will be properly constructed to represent the triangle ABD in fig. 66. Wherefore, if we measure ABD, we shall find the required angle.

Ex. 5. If $3W = 4P$, find the angle ABD.

Ex. 6. If $W = 2P$, find the angle ABD.

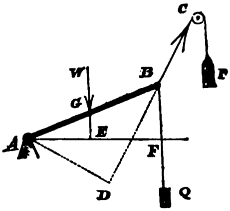
Ex. 7. If $W = 3P$, why cannot the angle ABD be found?

Problem IX.

AF, fig. 68, is a horizontal line, AB is a beam having a hinge at A, and supported in an inclined position by a string ABC, passing over a pulley C, and having a weight P suspended; also, another weight Q is suspended from B, and the weight of the

beam is balanced by these two. It is required to express the condition of equilibrium.

Fig. 68.



Let a perpendicular AD be drawn from the fulcrum A to BD, the direction of the string CB produced; also, let the vertical lines drawn downwards from G and B, meet the horizontal AF at E and F. Then the *moment* of W is $W \times AE$, and that of Q is $Q \times AF$; and these two forces tend to turn the lever downwards about A. Also, the *moment* of the tension of CB, which tension is equal to P, is $P \times AD$, and this tends to turn the lever upwards about A.

Hence, by Prop. V., the total effect of these forces is the sum of the two downward moments, $W \times AE + Q \times AF$, deducting the upward moment, $P \times AD$. But if the lever be held at rest by the forces, this effect must be nothing: wherefore, the opposing moments must be just equal to each other, that is, we have

$$W \times AE + Q \times AF = P \times AD,$$

which is the condition of equilibrium required.

Ex. 1. If $W = 10$, $Q = 10$, angle $BAF = 30°$, angle $ABD = 30°$; find P.

Ex. 2. On the same supposition, only that angle $ABD = 60°$; find P.

Ex. 3. On the same supposition, only that angle $ABD = 45°$, and $Q = 20$; find P.

Ex. 4. If $Q = W$, $P = 3\,W$, and angle $BAF = 20°$; find angle ABD.

Ex. 5. If $Q = 2W$, $2P = 5W$, and angle $BAF = 60°$; find angle ABD.

Ex. 6. On the same supposition, only that angle ABD is given to be equal to $60°$; find angle BAF.

MATHEMATICAL SOLUTIONS OF THE ABOVE PROBLEMS.

Problem VIII. Let angle $ABD = \theta$; then, $AD = AB \sin.\ \theta$; wherefore,

$$P : W :: AG : AB \sin.\ \theta.$$

$$\text{Whence } P = \frac{W.AG}{AB \sin.\ \theta} = \frac{W}{2 \sin.\ \theta}.$$

This equation gives P in terms of W and θ, and by it we may solve any of the examples above given.

Problem IX. Let angle $ABD = \theta$, and angle $BAF = a$; then $AD = AB \sin.\ \theta$, $AF = AB \cos.\ a$, and $AE = AG \cos.\ a$: wherefore,

$$W.AG \cos.\ a + Q.AB \cos.\ a = P.AB \sin.\ \theta.$$

Whence, observing that $AB = 2AG$, we find

$$P = \frac{(W + 2Q) \cos.\ a}{2 \sin.\ \theta}.$$

This equation gives P in terms of W, Q, a, and θ, and by it any of the examples may be solved.

CHAPTER III.

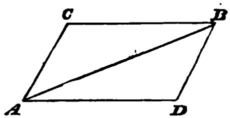
THE COMPOSITION OF FORCES ACTING AT THE SAME POINT.

(71.) ONE of the most important parts of mechanics is that which teaches how to find the resultant of two or more forces acting at the same point, for, by means of the rules for the *composition* of such forces, a considerable number of mechanical investigations are carried on. Indeed, this part of the subject is, properly speaking, the beginning and foundation of the whole range of the mechanical sciences, though, for the sake of making the student's progress more easy, we have introduced the principle of the concurrence of three forces, and the principle of the lever, previously. We shall in the present chapter enunciate and prove the celebrated rule, or principle, called the *parallelogram of forces*, which enables us to find the resultant of two forces, and thence, of any number of forces, acting at the same point. This rule may be deduced from the principle of the lever, and we shall show in what manner; but we shall also prove it independently of that principle, as most writers on the subject of mechanics consider that the parallelogram of forces should be made the foundation of the subject, and therefore be proved independently of any other principle.

PRINCIPLE OF THE PARALLELOGRAM OF FORCES.

If two forces, represented by the lines AC, AD, fig. 69, act at the same point A, and if a parallelogram $ACBD$ be formed upon AC and AD, the diagonal AB of that parallelogram will represent the resultant of the two forces.

Fig. 69.



(72.) In this enunciation the student will remember the meaning of the word *represent*, as explained at length in page 64. The forces represented by AC and AD, are forces acting in the directions in which these two lines are drawn, and containing each as many pounds as there are units in the lines AC and AD respectively; and the same may be said of the force represented by AB. Hence, according to the principle here stated, if we wish to find the resultant of two forces acting at the same point A, all we have to do is, to draw from A two lines, AC and AD, in the direction in which the two forces act, and measure them according to any convenient scale, (see page 68,) so that there shall be as many units in AC as there are pounds in the force acting in that direction, and as many units in AD as in the force in that direction. Then we must construct upon AC and AD the figure called a *parallelogram*, which we do by drawing CB parallel to AD, and DB parallel to AC, to meet at B; and, having drawn the diagonal AB, we must measure it and find how many units it contains. *Then the* resultant of the two forces, AC and AD, *is a* force of as many pounds as

there are units in AB, and it acts in the direction in which AB is drawn.

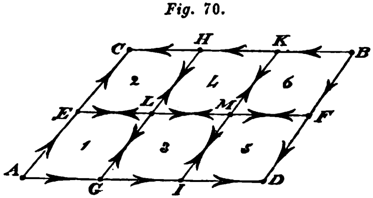
We shall prove the truth of the principle of the parallelogram of forces, in the following two propositions, of which one shows that the *direction* of the resultant is that in which AB is drawn, and the other, that the *magnitude* of the resultant is represented by the length of AB, that is, that there are as many pounds in the resultant as there are units in AB.

Proposition VI.*

The diagonal AB of the parallelogram $ACBD$, represents the direction *in which the resultant of the forces AC and AD acts.*

Let $ACBD$, fig. 70, represent a rigid body in the shape of a parallelogram, and let us suppose that the side AC contains a certain number of units, say two, viz. AE and EC, and that the side AD contains, say three units, viz. AG, GI, and ID. Draw the lines HG and KI parallel to AC, and the line EF parallel to AD, crossing the two former lines at L and M. The parallelogram is thus divided into 6 small parallelograms, 1, 2, 3, 4, 5, 6, and every side of each of these parallelograms is unity.

Fig. 70.



Let us suppose that four forces, each a pound, act along the sides of parallelogram 1, two at A, and two at L, as is represented in the figure.

* *For a different proof of this proposition, founded on the principle of the lever, see page 141.*

Let us also suppose that four such forces act along the sides of parallelogram 2, two at E, and two at H, as is shown in the figure. And likewise, along the sides of each of the parallelograms, 3, 4, 5, and 6, let us suppose that four forces, each a pound, act, two at G and two at M, two at L and two at K, two at I and two at F, two at M and two at B; as is shown in the figure.

Now, the diagonal AL of parallelogram 1 bisects the angle EAG, and the angle ELG also, because the sides of the parallelogram are all equal;* and the resultant of the two equal forces at A also bisects the angle EAG, by Axiom VIII.; therefore the resultant of the two forces at A acts along the line AL. In like manner, it may be shown that the resultant of the two forces at L, (acting along LE and LG,) acts along the line LA. These two resultants, therefore, act in opposite directions, and they are manifestly equal, because the forces at A and at L are equally inclined to each other. Hence, by Axiom IX., the two resultants, and therefore the two pairs of forces at A and L, balance each other.

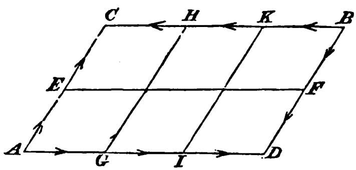
In like manner it may be shown, that the four forces acting along the sides of parallelogram 2, two at E and two at H, balance each other, and the same may be said of the other parallelograms 3, 4, 5, and 6. It appears, then, that the whole set of forces represented in the figure balance each other, and that the whole rigid parallelogram under their action has no tendency to move.

* This may be shown by Euclid, Book i. or by measurement, if the student does not know anything of Euclid. By Euclid, Book i. $\angle EAL = \angle ELA$, because $EA = EL$; also, $\angle ELA = \angle GAL$, because AE is parallel to GL: therefore $\angle EAL = \angle GAL$.

But the forces acting along the lines *EF*, *GH*, *IK*, all balance each other, for they consist of pairs of equal and opposite forces, as is evident by inspecting the figure; for instance, along *EF* there are a pair of equal and opposite forces acting on the line *EL*, another pair on the line *LM*, and another pair on the line *MF*. All these pairs of forces, since they balance each other, may, by Axiom II. be removed, and then there remain only the forces acting along *AC*, *AD*, *BC*, and *BD*, as is shown in fig. 71.

It appears, then, that the forces represented in fig. 71, balance each other without the assistance of the forces on the lines *EF*, *HG* and *KI* represented in fig. 70. The rigid parallelogram, therefore, has no tendency to move when the forces shown in fig. 71 act upon it, and by the principle of the transmission of force through a rigid body, we may suppose the forces along *AC* to act all at *A*; those along *AD* at *A*, those along *BC* at *B*, and those along *BD* at *B*.

Fig. 71.

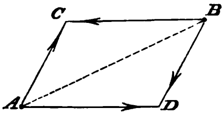


Now, these forces are *represented* by the sides of the parallelogram; for, along *AC*, which is 2 units long, we have 2 pounds acting; along *AD*, which is 3 units long, we have 3 pounds acting; along *BD*, which is 2 units, we have 2 pounds; and along *BC*, which is 3 units, we have 3 pounds. We have proved this to be true on the supposition that the sides of the parallelogram are 2 and 3 *respectively*; but if they had been 4 and 5, 6 and

11, or any other numbers whatever, it is manifest from the nature of the case, that the same conclusion and result would have followed.

Hence, we conclude in general, that if *ACBD*, fig. 72, be a rigid body in the shape of a parallelogram, and if four forces represented by its sides act upon it, two, namely *AC* and *AD*, at *A*, and the other two, namely *BC* and *BD*, at *B*, the parallelogram will not have any tendency to move.

Fig. 72.



Let *R* represent the resultant of *AC* and *AD*, and *S* that of *BC* and *BD*, then, since the forces *AC* and *AD* balance the forces *BC* and *BD*, it follows that *R* must balance *S*. Therefore, by Axiom XII., *R* and *S* must be exactly equal and opposite, which cannot possibly be the case except both act along the line *AB*.

It follows then that *R*, the resultant of the forces *AC* and *AD*, acts along the line *AB*; in other words, the resultant of the two forces represented by *AC* and *AD* *acts in the direction represented by AB*. *Which was to be proved.*

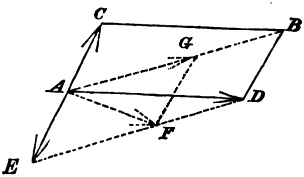
PROPOSITION VII.

The diagonal AB represents also the magnitude *of the resultant of the forces represented by AC and AD.*

If *AB* does not represent the resultant in magnitude, suppose, if possible, that some other length, *AG*, fig. 73, does; draw *DE* parallel to *AB*, to meet *CA* produced in *E*, thus forming the parallelogram *ABDE*; draw *GF* parallel to *AE*,

and join AF: AE is evidently equal to AC, for both are equal to BD.

Fig. 73.



Then, let us suppose that the three forces represented by AC, AD, and AE, act at A. Of these, the two forces AD and AC are equivalent to their resultant AG; therefore, the three forces AD, AC, and AE, produce the same effect as AG and AE together; but these two forces also produce the same effect as their resultant, which, by Prop. VI., must act along the diagonal (AF) of the parallelogram $AGFE$; therefore the forces AD, AC, and AE together produce the same effect as a force acting along AF.

Now we may consider the three forces AC, AD, and AE somewhat differently; for AC and AE, being equal and opposite forces, may be removed; therefore AD alone must produce the same effect as the three forces, that is, as we have proved, the same effect as a force acting along AF. But this cannot be true unless AF and AD lie in the same direction with each other, in which case the points F and D, and therefore the points G and B, must coincide, so that AG and AB must be one and the same line.

Hence AG must be the same length as AB, and therefore the magnitude of the resultant of AC and AD is represented by AB. *Which was to be proved.*

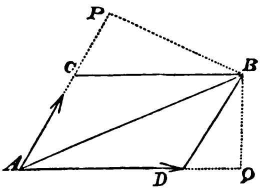
We have now completely proved the principle of the parallelogram of forces, by independent

reasoning. To deduce it from the principle of the lever, we must substitute for the demonstration of Proposition VI. above given, the following, depending on Proposition III.

PROPOSITION VI. (*otherwise proved.*)

Let $ACBD$, fig. 74, be supposed to be a lever in the shape of a parallelogram, B being the fulcrum, and suppose that two forces represented by AC and AD act upon it. Draw BP and BQ from the fulcrum, upon the directions of the two forces produced, and join A and B.

Fig. 74.



Then, by the first and second books of Euclid, the area of the triangle ABD is equal to $AD \times BQ$, and the area of the triangle ACB is equal to $AC \times BP$; also, the two triangles are equal to each other: therefore we have

$$AC \times PB = AD \times BQ.$$

And therefore, by the rule of proportion,

$$AC : AD :: BQ : BP.$$

Hence the forces are to each other inversely as the perpendiculars upon them from the fulcrum, and therefore they balance each other, by Proposition III.

Now, if the two forces balance each other, their resultant cannot have any tendency to turn the lever about the fulcrum, therefore it must act along the line AB, for if it fall on one side or the other of this line, it would not pass through the

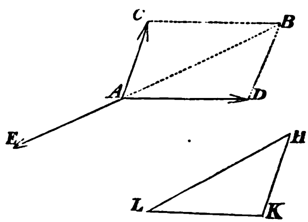
fulcrum, and would be therefore sure to turn the lever about the fulcrum.

Hence the resultant of *A C* and *A D* acts in the direction represented by the diagonal *A B*. *Which was to be proved.*

Proposition VIII.

If three forces, acting at the same point, balance each other, they must be proportional to the sides of the triangle formed by any three lines parallel to their directions.

Fig. 75.



Let *A C*, *A D*, and *A E*, fig. 75, represent the three forces; draw *C B* parallel to *A D*, and *D B* to *A C*, and join *A* and *B*. Then, since *A E* balances *A C* and *A D*, *A E* must balance the resultant of *A C* and *A D*; but *A B* represents that resultant by parallelogram of forces; therefore *A E* balances *A B*, and consequently *A E* and *A B* must be equal and opposite, (Axiom XII.) wherefore *A E* and *A B* lie in the same straight line, and are of the same length.

Hence the sides of the triangle *A B D* represent the magnitudes of the three balancing forces, *A C*, *A D*, and *A E*; for *B D* is equal in length to *A C*, and *A B*, as we have proved, is equal to *A E*.

Now form a triangle *H K L* by drawing any three lines, *H K L*, parallel to the three forces, *A C*, *A D*, *A E*; or, what is the same thing, parallel to the sides *B D*, *D A*, *A B*, of the triangle *A B D*. Then the triangle *H K L* will be exactly *similar in*

rm to the triangle *ABD*, differing from it only in ze;* in fact, *HKL* will be a copy, either enrged or diminished as the case may be, of the angle *ABD;* so that the corresponding sides of e two triangles will be proportional to each her. For example, if *AD* be twice *BD*, and *B* three times *BD*, *KL* will be twice *HK*, d *LH* will be three times *HK*.

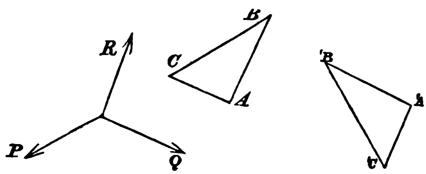
Hence, since *BD*, *AD*, and *AB* represent the agnitudes of the three balancing forces, and nce these lines are proportional to *HK*, *KL*, d *LH*, respectively; it follows, that the three lancing forces are proportional to the sides of e triangle *HKL* formed by any three lines awn parallel to the forces.

Observe, each force is proportional to *that side* of e triangle which is parallel to it. Thus *AC* is oportional to *HK*, *AD* to *KL*, and *AE* to *LH*.

Corollary 1.—*If a triangle be formed by three lines right angles to the directions of three forces which lance each other, the forces are respectively prortional to the sides of the triangle so formed.*

If we draw lines *AB*, *BC*, and *CA*, fig. 76,

Fig. 76.



rallel to the forces *R*, *P*, and *Q*, which balance ch other, the sides of the triangle so formed are

* The student who has read the Sixth Book of Euclid will derstand clearly what is meant by *similar triangles* without e explanation.

proportional to the forces respectively. Now let $A'B'C'$ be a triangle of exactly the same shape and size as ABC, only in a different position, being turned round through 90° from the position of ABC; then $A'B'$ will be at right angles to AB, $A'C'$ to AC, and $B'C'$ to BC. Also, the sides of the two triangles being equal to each other, the forces, which are proportional to the sides of ABC, must also be proportional to the sides of $A'B'C'$.

Now $A'C'B'$ is a triangle formed by lines at right angles to the forces, and its sides are proportional to the forces; namely, each side is proportional to the force to which it is perpendicular.

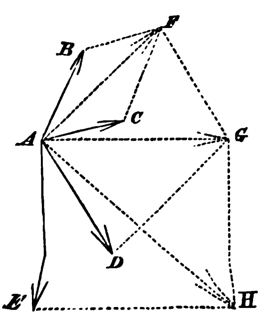
Corollary 2.—The same would be true if $A'B'C'$ had been turned round through any other angle besides 90° from the position of ABC.

This is a very important proposition in the solution of problems.

Proposition IX.

To find the resultant of any number of forces acting at the same point.

Fig. 77.

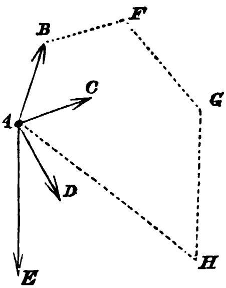


Let AB, AC, AD, and AE, fig. 77, represent the forces; on AB and AC form the parallelogram $ABFC$, and draw the diagonal AF; on AF and AD form the parallelogram $AFGD$, and draw the diagonal AG; on AG and AE form the parallelogram $AGHE$, and draw the diagonal AH. Then AH represents the resultant required.

For, by the Parallelogram of Forces, the force AH is equivalent to the forces AE and AG; therefore, since AG is for the same reason equivalent to AD and AF, AH is equivalent to AE, AD, and AF; therefore, since AF is in like manner equivalent to AC and AB, AH is equivalent to AE, AD, AC, and AB; in other words, AH is the resultant required. *Which was to be done.*

It will not be necessary to draw all the lines in the figure in order to find the resultant AH; it will be sufficient, as in fig. 78, to draw BF parallel and equal to AC, FG parallel and equal to AD, GH parallel and equal to AE, and then, by joining the points A and H, we find the resultant AH: for this evidently comes to the same thing, as if we constructed each of the parallelograms as above, and so found AH.

Fig. 78.



This proposition, showing how to determine the resultant of a set of forces by means of a polygon, is often called the *Polygon of Forces.*

Proposition X.

To determine whether a given set of forces applied to a point are in equilibrium or not.

Let AB, AC, AD, and AE be the given forces (fig. 78, previous Proposition), and find their resultant (AH) in the manner just explained; then the given forces *produce* the same effect as AH,

and therefore, if AH turns out to be nothing, *i. e.* if the point H coincides with A, the forces produce no effect; or, in other words, they balance each other. Hence the forces will be in equilibrium, if the point H coincides with A; otherwise they will not. *Which was to be determined.*

MATHEMATICAL FORMULÆ.

Before we proceed to give examples and problems illustrative of the propositions just proved, we shall briefly deduce the mathematical formulæ which are applicable in the composition and resolution of forces. This is intended of course only for those students who have some acquaintance with Algebra and Trigonometry.

Proposition XI.

To express the parallelogram of forces, and the condition of equilibrium of three forces acting at the same point, by mathematical formulæ. (*Page* 135.)

Let us represent the two forces AD and AC by P and Q, and the resultant AB by R; also, let a denote the angle CAD which P and Q make with each other; then, referring to fig. 69, we have, in the triangle ABD,

$$DA = P,\ BD = AC = Q,\ AB = R,$$
$$\angle ADB = 180° - \angle CAD = 180° - a.$$

Now, by trigonometry, we have

$$AB^2 = AD^2 + BD^2 - 2\,AD\,.\,BD \cos.\ ADB.$$

Whence, since $\cos. (180° - a) = -\cos. a$, we find

$$R^2 = P^2 + Q^2 + 2\,P\,Q \cos. a \quad \ldots\ldots \quad (1.)$$

This formula gives R in terms of P, Q, and a, and, in fact, *expresses the parallelogram of forces mathematically.*

Again, referring to Proposition VIII. p. 142, fig. 75, we have, by trigonometry, from the triangle ABD,

$$\frac{AB}{\sin. ADB} = \frac{BD}{\sin. BAD} = \frac{AD}{\sin. ABD}.$$

Now, $\angle ADB = 180° - \angle CAD$, $\angle BAD = 180° - \angle EAD$.

And $\angle ABD = \angle CAB = 180° - \angle CAE$.

Wherefore we find, observing that $BD = AC$, and $AB = AE$,

$$\frac{AE}{\sin. CAD} = \frac{AC}{\sin. EAD} = \frac{AD}{\sin. CAE} \quad . \; . \; . \quad (2.)$$

In other words, the three forces AE, AC, AD, which balance each other, are respectively proportional to the sines of the angles they make with each other; that is to say, each force is proportional to the sine of the angle made by the other two forces; AE to the sine of the angle made by AC and AD, AC to the sine of the angle made by AE and AD, and AD to the sine of the angle made by AE and AC.

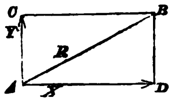
This is the *condition of equilibrium of three forces acting at the same point expressed mathematically. Which was to be done.*

Proposition XII.

To find the formulæ for resolving and compounding forces rectangularly.

Let X and Y be two forces represented by $A\,D$ and $A\,C$, fig. 79, the angle $C\,A\,D$ being a right angle, and let R be the resultant of X and Y, which will of course be represented by the diagonal $A\,B$ of the rectangle $A\,C\,B\,D$; also, let θ denote the angle $B\,A\,D$, which R makes with X. Then we have, (observing that $BD = A\,C = Y$, $A\,D = X$, and $A\,B = R$,)

Fig. 79.



$$\left.\begin{array}{l} X = R \cos.\ \theta,\ Y = R \sin.\ \theta \\ R^2 = X^2 + Y^2 \\ \tan.\ \theta = \dfrac{Y}{X} \end{array}\right\} \ldots (3.)$$

These formulæ give X and Y in terms of R and θ, R in terms of X and Y, and θ in terms of X and Y.

Now, X and Y are called *rectangular components* of R, and when we find X and Y in terms of R and θ, we *resolve* R into rectangular components, and if we find R and θ in terms of X and Y, we *compound* the rectangular forces X and Y into their resultant R. *Which was to be done.*

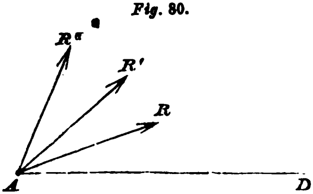
Proposition XIII.

To determine mathematically the resultant of a given set of forces acting at the same point, and the conditions of their equilibrium.

Let the forces be represented by R, R', R'', &c., and suppose that they make angles θ, θ', θ'', &c. with a given line $A\,D$, fig. 80. Resolve *these forces* rectangularly into forces acting along

and at right angles to $A D$; then the forces R, R', R'', &c. will, by the preceding proposition, be equivalent to the following acting along $A D$, viz.:—

Fig. 30.



$$R \cos. \theta,\ R' \cos. \theta',\ R'' \cos. \theta'',\ \&c.;$$

together with the following acting at right angles to $A D$, viz.:—

$$R \sin. \theta,\ R' \sin. \theta',\ R'' \sin. \theta'',\ \&c.$$

Wherefore, the whole set of forces, R, R', R'', &c., are equivalent to the two total forces, which we shall for brevity denote by $X_{,}$ and $Y_{,}$, namely,

$X_{,} = R \cos. \theta + R' \cos. \theta' + R'' \cos. \theta'' +$ &c. acting along $A D$.

And $Y_{,} = R \sin. \theta + R' \sin. \theta' + R'' \sin. \theta'' +$ &c. acting at right angles to $A D$.

Let $R_{,}$ be the resultant of $X_{,}$ and $Y_{,}$, and therefore of R, R', R'', &c.; and let $\theta_{,}$ denote the angle which $R_{,}$ makes with $A D$; then, by Prop. XII.,

$$\left.\begin{aligned} R_{,}^2 &= X_{,}^2 + Y_{,}^2 \\ \tan. \theta_{,} &= \frac{Y_{,}}{X_{,}} \end{aligned}\right\} \ldots\ldots (4.)$$

The first of these equations gives $R_{,}$ the resultant of the forces, and the second gives the angle $\theta_{,}$ at which $R_{,}$ is inclined to $A D$. Thus the *resultant is determined* completely.

If the forces be in equilibrium, their resultant R must be zero, and therefore we have

$$X_{,}^{2} + Y_{,}^{2} = 0;$$

which, since squares are always positive, and cannot therefore cancel each other, requires that $X_{,}$ and $Y_{,}$ shall be *each* of them equal to zero. We have therefore, putting for $X_{,}$ and $Y_{,}$ their values,

$$\left.\begin{array}{l} R\cos.\theta + R'\cos.\theta' + R''\cos.\theta'' + \&c. = 0 \\ R\sin.\theta + R'\sin.\theta' + R''\sin.\theta'' + \&c. = 0 \end{array}\right\} . \quad (5.)$$

These equations express the two conditions necessary for the equilibrium of the forces R, R', R'', &c. *Which was to be done.*

This proposition is nothing more than Propositions IX. and X. expressed mathematically.

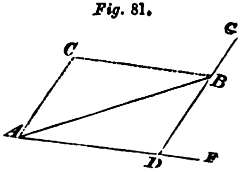
PROBLEMS AND EXAMPLES.

Examples of the Parallelogram of Forces.

Simplified Construction.—When we employ the parallelogram of forces graphically, it is only necessary to draw the triangle $A B D$, fig. 69; for the two forces, and their resultant, and the angles which they make with each other are all contained in this triangle. Suppose then that the two forces and the angle which they make with each other are given, and it is required to find the resultant graphically; we may proceed as follows:—

Draw a line $A F$, fig. 81, measure upon it a portion $A D$, containing as many units as there are pounds in one of the given forces; draw $D G$, making the angle $G D F$ equal to the given angle

Fig. 81.



at which the two forces are inclined to each other, and measure upon DF a portion DB, containing as many units as there are pounds in the other force. Then join A and B, and AB will represent the required resultant; so that, by measuring the number of units in AB, we shall know how many pounds there are in the resultant; and by measuring the angle BAD, we shall know at what angle the resultant is inclined to one of the forces.

That this is so may be shown by completing the parallelogram $ACBD$ by drawing the dotted lines AC and CB; for it is evident that AC and AD are the two given forces, and that they make the proper angle CAD with each other; AC is equal to BD, and the angle CAD is equal to the angle GDF, because $ACBD$ is a parallelogram. It is clear then that AB thus constructed is the proper resultant.

The following examples may be easily constructed by the triangle ABD.

Ex. 1.—Find the resultant of the forces 3 lbs. and 4 lbs. when they are inclined to each other at an angle of 90°.

Ex. 2.—Find the resultant of the same forces inclined at an angle of 45° to each other.

Ex. 3.—Find the resultant of the same forces inclined at an angle of 135° to each other.

Ex. 4.—What is the least and what is the greatest resultant that two forces, 8 lbs. and 24 lbs., can have?

Ex. 5.—At what angle do the forces 4 and 7 act when their resultant is 9?

In this case the three sides of the triangle ABD, fig. 81, are given, viz. $AD=7$, $BD=4$, $AB=9$; and it is required to find the angle BDF, which, as we have shown, is equal to the angle at which the two forces act. To do this, draw AF, and measure a portion AD on it equal to 7; with centre A and radius 9 describe an arc of a circle, and also with centre D and radius 4 describe another arc of a circle. Then the point where these two arcs intersect will be the point B, and we have only to draw the line BD, and measure the angle BDF, and we shall so find the angle at which the two forces act.

This is a case where three sides of a triangle are given, and it is required to construct the triangle and measure one of its angles, or rather one of what are called its *exterior* angles.

Ex. 6.—At what angle must the forces 6 and 8 act in order that their resultant may be 10?

Ex. 7.—Two forces, one of which is 7, and the other unknown, act at an angle of 60°; what must the unknown force be, in order that the resultant of the two may be inclined at an angle of 45° to the known force?

In this case AD, fig. 81, is 7, $\angle BDF = 60°$, and $\angle BAD = 45°$; it is required to find DB.

Ex. 8.—On the same supposition, what must the unknown force be, in order that the resultant may be inclined at an angle of 30° to the known force?

Ex. 9.—On the same supposition, what must the unknown force be, in order that the resultant may be 14?

Ex. 10.—On the same supposition, what must

the resultant be, in order that it may be inclined at an angle of 20° to the unknown force?

Ex. 11.—The resultant of two forces is 10, and it makes an angle of 30° with one of the forces, and of 90° with the other; find the two forces.

Ex. 12.—Resolve a force 10 into two other forces at right angles to each other, one of them making an angle of 60° with the force 10.

Ex. 13.—Resolve a force 12 into two others at right angles to each other, and both equally inclined to the force 12.

Ex. 14.—A force 10 acts obliquely at an angle of 50° to the horizon; resolve it into two forces, one horizontal and the other vertical.

Ex. 15.—The horizontal and vertical components of a force are 6 and 3; what is the force, and at what angle is it inclined to the horizon?

When we speak, as we do here, of the *horizontal and vertical components* of a force, we mean the two forces into which it may be resolved, of which one acts horizontally and the other vertically. When a force is resolved horizontally and vertically in this manner, the horizontal component is often called the *horizontal effect of the force,* and the vertical, the *vertical effect.*

Ex. 16.—What horizontal effect does a force 10, inclined at 60° to the horizon, produce?

Ex. 17.—The horizontal component of a force is double its vertical component; at what angle to the horizon is the force inclined?

Ex. 18.—The vertical component of a force is half the force; at what angle to the horizon is the force inclined?

Examples of the Polygon of Forces. (*Prop. IX.*)

Ex. 1.—If the forces AB, AC, AD, and AE, fig. 77, be respectively 4, 6, 8, and 10, and if the angles BAC, CAD, and DAE, be respectively 90°, 60°, and 30°, find the resultant.

Ex. 2.—Supposing that there are but three forces, $AB = 10$, $AC = 10$, $AD = 8$, and that the angles BAC and CAD are each 120°; find the resultant.

Ex. 3.—Making the same supposition, except that $AC = 8$ and $AD = 4$; find the resultant.

Ex. 4.—Making the same supposition as in Ex. 2, except that AD is unknown; find what AD must be, in order that the resultant may be 5.

Ex. 5.—If the angles BAC and CAD be each 45°, and the forces each 10, find the resultant.

Ex. 6.—Making the same supposition, except that AD is unknown; find what AD must be, in order that the resultant may be inclined at an angle of 60° to AB.

Ex. 7.—On the same supposition, what is AD when the resultant acts in the same direction as AC?

Examples of the Equilibrium of Forces.
(*Props. VIII. and X.*)

Ex. 1.—Three forces, AC, AD, AE, (fig. 75,) balance each other, the angles CAD and DAE being 90° and 40°, and AC is 10; find AD and AE.

Ex. 2.—Three forces, 4, 8, and 12, balance each other; at what angles are they inclined to each other?

Ex. 3.—What force will balance the forces 6 and 10 acting at an angle of 60°?

Ex. 4.—Four forces, as in fig. 77, balance each other, of which $AB = 10$, $AC = 10$, $AD = 15$, and $AE = 20$; also $\angle BAC = 120°$: find the angles CAD and DAE.

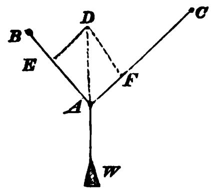
Ex. 5.—Making the same supposition, except that AE is unknown, and CAD is 60°; find AE.

PROBLEM X.

If a weight W be suspended from two points by the strings AB and AC, fig. 82, to determine the tensions on the strings.

Let the vertical line DA represent the weight, and draw the lines DE and DF parallel to the strings. Then the force DA is equivalent to the two forces represented by EA and FA, which are therefore the tensions on the strings, as required.

Fig. 82.



Simplified Construction.—Draw only the lines AD, AE, and DE; observing that EAD and EDA are equal to the angles which the strings make with the vertical.

Ex. 1.—$W = 10$, AB and AC are inclined at angles of 30° and 60° respectively to the vertical; find the tensions.

Ex. 2.—CB is a horizontal line and equal to 4, $AB = 3$, $AC = 2$, and $W = 10$; find the tensions.

Ex. 3.—*CB is horizontal,* and $AC = AB$; find

the angle CAB, so that the tension on each string may be twice the weight.

Ex. 4.—The strings are not strong enough to bear a tension of more than $6W$ each; find how great the angle CAB may be without their breaking.

Ex. 5.—If BAC be one string, A being a little smooth pulley from which W is suspended; find the tensions when $W=10$, and $\angle BAC=45°$.

In this case the tensions on each side of the pulley must be equal, because, if one were greater than the other, the pulley being perfectly smooth and capable of turning freely, would be set in motion by the preponderating tension.

Ex. 6.—On the same supposition, if the tension be 15 on each string, and $\angle BAC=60°$; find W.

Problem XI.

In the case supposed in Prob. I. p. 95, the weight of the beam being given, to find the pressure the beam exercises against the wall at A and on the prop at C.

The forces which keep the beam at rest being its weight, the reaction of the wall at A, and the reaction of the prop at C, fig. 46, these forces, as we have shown, must meet in the point H; and therefore we have the case of three forces, balancing each other, acting at the same point. Now the sides of the triangle CLH, fig. 48, are parallel to the directions of these forces, namely, CH to the weight of the beam, LH to the reaction of the wall, and CL to that of the prop. Wherefore by Proposition VIII. the forces are proportional to the corresponding sides of the triangle. If, therefore, W denote the weight of the beam, R the

reaction of the wall, and P that of the prop; we have,

$$R : W :: LH : CH;$$

$$P : W :: CL : CH.$$

If W be given, and if CL, HC and LH be determined by measurement, we may find R and P from these proportions by the Rule of Proportion. Now R is equal and opposite to the pressure of the beam against the wall, and P to the pressure of the beam upon the prop; consequently these two pressures are determined, as required.

Observe, $\angle LCH$ is the complement of the angle of inclination of the beam to the vertical, because CL is at right angles to AC.

Ex. 1.—If $W = 10$, and the beam is inclined at $60°$ to the vertical; find R and P.

Ex. 2.—If $W = 10$, and the beam is inclined at $30°$ to the vertical; find R and P.

Ex. 3.—If the beam presses against the wall with a force equal to double its weight; find the inclination of the beam.

Ex. 4.—If the pressure against the wall is half the pressure upon the prop; find the inclination of the beam.

Problem XII.

In the case supposed in Problem II. the weight being given, find the pressure of the beam against the wall and upon the prop. (*Page* 99.)

The line drawn from A at right angles to the wall, the line from C at right angles to the beam, and the vertical line through G,

form a triangle $C L H$, whose sides are proportional to R, P and W, as in the preceding proposition, and we have therefore the same proportions, viz.

$$R : W :: L H : C H;$$

$$P : W :: C L : C H;$$

by which the problem may be solved.

Observe, in the triangle $C L H$, $\angle H C L$ is the complement of the inclination of the beam to the vertical, and $\angle C H L$ is the complement of the inclination of the wall to the vertical.

Ex. 1.—$W = 10$, the wall makes an angle of 20°, and the beam an angle 20° also with the vertical; find P and R.

Ex. 2.—The weight of the beam, the pressure against the wall, and the pressure upon the prop, are all equal to each other; find the inclination of the beam and that of the wall.

Ex. 3.—Find the same when $P = R = 3\,W$.

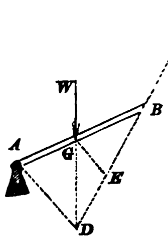
Ex. 4.—When the beam rests in a horizontal position, what is the amount of the pressure against the wall, and of that on the prop?

Problem XIII.

If $A B$, fig. 83, be a beam, one end of which, A, is kept fixed by a hinge or fulcrum, and the other end, B, supported by a string $B C$ fastened to the fixed point C; to find the magnitude and direction of the pressure which the beam exercises on the fulcrum A.

Draw $G D$ through the middle point G of the beam to meet the direction of the string produced at D, join $A D$, and draw $G E$ parallel to

Fig. 83.



AD. Then the forces which keep the beam at rest are, its weight, the tension of the string, and the reaction of the fulcrum; of which forces, the former two meet at D. Wherefore the reaction of the fulcrum must also pass through D, and must therefore be a force acting in the direction of the line DA. Let P represent this force, and W the weight of the beam: then, in the triangle GDE, GD is parallel to W, GE to P, and DE to the tension of the string; therefore, by Proposition VIII., we have,

$$P : W :: GE : GD.$$

Which proportion, when W is known, and GE and GD determined by measurement, will give P, as required. The pressure of the beam on the fulcrum is of course equal, and opposite to P.

It is not necessary to draw the line GE; for, since G is the middle point of AB, GE is half of AD: wherefore, to find GE we have only to measure AD and take half of it.

Ex. 1.—The beam is horizontal, and the string is inclined at 30° to the vertical; find P.

Ex. 2.—The beam is inclined at 30° to the horizon, and the string at 45° to the vertical; find P.

Ex. 3.—The beam is horizontal, and P is half of W; what is the inclination of the string?

Ex. 4.—Find the same, on same supposition, except that $P = W$.

Ex. 5.—$P = W$, and the string makes an angle

of 45° with the vertical; find the inclination of the beam.

MATHEMATICAL SOLUTIONS.

The *Examples* of the Parallelogram of Forces, and those that follow, may all be solved by the formulæ given in Propositions XI. XII. and XIII.

Problem X. The same may be said of this problem.

Problem XI. If θ be the angle which the beam makes with the horizon, we have,

$$LH : CH :: \sin.\,\theta : \cos.\,\theta;$$

$$CL : CH :: \quad 1 : \cos.\,\theta.$$

Wherefore the proportions obtained become,

$$R : W :: \sin.\,\theta : \cos.\,\theta, \text{ and } \therefore R = W \tan.\,\theta;$$

$$P : W :: \quad 1 : \cos.\,\theta, \text{ and } \therefore P = W \sec.\,\theta.$$

Problem XII. In this case, if a be the angle at which the wall is inclined to the horizon, and if we assume θ to be the angle at which the beam is inclined to the horizon, we have, in the triangle LHC, $\angle LCH = \theta$, $\angle CHL = a$; and therefore,

$$R : W :: LH : CH :: \sin.\,\theta : \sin.\,(a + \theta);$$

$$P : W :: CL : CH :: \sin.\,a : \sin.\,(a + \theta).$$

Wherefore

$$R = \frac{\sin.\,\theta}{\sin.\,(a + \theta)} W, \quad P = \frac{\sin.\,a}{\sin.\,(a + \theta)} W.$$

Problem XIII. Let $\angle AGD$, the angle at which the beam is inclined to the vertical, be denoted by

a, and $\angle\, GDB$, the angle at which the string is inclined to the vertical, by θ: then,

$$P : W :: GE : GD :: \tfrac{1}{2} AD : GD.$$

Therefore $P = W \dfrac{AD}{2\,GD}$.

Now in the triangle DGB, if we denote GD by x, we have,

$$GB : x :: \sin.\ \theta : \sin.\ (a - \theta);$$

$$\therefore\ GB = x \frac{\sin.\ \theta}{\sin.\ (a-\theta)},\ \text{and } AG = GB.$$

Also in the triangle AGD,

$$AD^2 = AG^2 + GD^2 - 2\,AG,\ GD \cos.\ a;$$

$$\text{or, } AD^2 = x^2\left(\frac{\sin.\ \theta}{\sin.\ (a-\theta)}\right)^2 + x^2 - 2\,x^2\,\frac{\sin.\ \theta \cos.\ a}{\sin.\ a - \theta}.$$

Hence,

$$P = W.\frac{AD}{2\,GD} = \frac{W}{2}\sqrt{\left(\frac{\sin.\ \theta}{\sin.(a-\theta)}\right)^2 + 1 - \frac{2\sin.\ \theta\cos.\ a.}{\sin.\ (a-\theta)}}$$

METHOD OF SOLUTION BY RESOLVING FORCES RECTANGULARLY.

In a great many cases the mathematical solution of mechanical problems is considerably facilitated by making use of the conditions of equilibrium obtained in Proposition XIII., namely, the two equations numbered (5). These conditions may be stated thus:—

If a set of forces acting at the same point balance each other, and if we find the rectangular components

of each force, by resolving each force into two others, one of which acts in a certain direction, which we may call the Primary Direction, and the other at right angles to that direction; then the sum of all the components in the primary direction must be zero, and the sum of those at right angles to the primary direction must also be zero.

Observe that the rectangular components of any force R, which makes an angle θ with the primary direction, are,

R cos. θ along the primary direction;

R sin. θ at right angles to it.

Also in adding together the components, say for instance those in the direction $A\,D$, we must give each component *its proper sign*, according as it tends in the primary direction, or in the opposite; that is, we must consider all the components which tend in the primary direction as positive, and those in the opposite direction as negative.

Bearing these observations in mind, we may solve the preceding problems in the following manner.

Problem XI. Taking the primary direction to be horizontal, the force R acts in that direction, W at right angles to it, and therefore R has no component at right angles to the primary direction, nor has W a component in that direction: also P makes an angle $90° - \theta$ with the primary direction; the components of P are therefore, P cos. $(90° - \theta)$, or P sin. θ, horizontally; and P. sin. $(90° - \theta)$, or P cos. θ, vertically. Hence we have,

horizontal components . . $R - P$ sin. $\theta = 0$;

vertical components . . . . $- W + P$ cos. $\theta = 0$.

Here we put P sin. θ negative, because we assume the direction of R to be the positive direction, and it is clear that the horizontal effect of P is contrary to R. Also we put W negative because we assume the upward to be the positive direction.

Problem XIII. Denoting the tension on the string by T, the horizontal components of the forces are, P sin. ϕ (if $\angle\, GDA = \phi$), and T sin. θ; and the vertical components are W, P cos. ϕ, and T cos. θ; observing, that $90^\circ - \theta$ and $90^\circ - \phi$ are the angles which T and P make with the horizon. Hence, giving the components their proper signs, viz. P sin. ϕ and W negative, and the others positive, we find

$$-P \sin. \phi + T \cos. \theta = 0$$

$$-W + P \cos. \phi + T \sin. \theta = 0$$

From these equations we may find P by eliminating θ, but we must also find ϕ, which may be done thus:—

$AG : GD :: \sin. \phi : \sin. (\alpha + \phi)$ $(\angle\, AGD = \alpha)$;

$BG : GD :: \sin. \theta : \sin. (\alpha - \theta)$.

Wherefore $\dfrac{\sin (\alpha + \phi)}{\sin. \phi} = \dfrac{\sin. (\alpha - \theta)}{\sin. \theta}$;

from which ϕ may be found in the terms of α and θ.

CHAPTER IV.

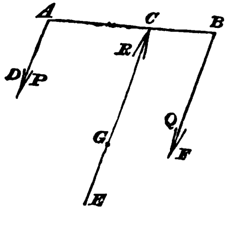
COMPOSITION AND EQUILIBRIUM OF FORCES ACTING ON A RIGID BODY AT DIFFERENT POINTS.

HAVING in the former chapter explained the method of compounding forces which act *at the same point,* we shall proceed to consider forces acting at *different points* of a rigid body, and investigate the rules for finding their resultant, and the conditions of their equilibrium. The simplest case of such forces is when they are parallel to each other, and we shall commence with this case.

COMPOSITION AND EQUILIBRIUM OF PARALLEL FORCES.

PROPOSITION XIV.

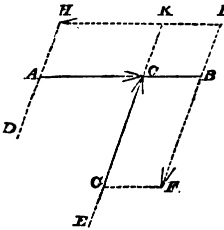
Fig. 84.



If AD, BF, *and* EC, *fig.* 84, *be three parallel forces, acting on a rod* AB, *or on a rigid body, they will balance each other when they are represented in magnitude by the lines* BC, AC, *and* AB *respectively; or, what is the same thing, when* AD, BF, *and* EC *are proportional to* BC, AC, *and* AB *respectively.*

This may be proved immediately from the Principle of the Lever stated in Proposition I., as we shall show in p. 166; but as many persons object to that principle as the foundation of Statics, and prefer the Parallelogram of Forces, we shall deduce the present proposition from the latter principle, or rather, from Proposition VI.

Fig. 85.



Let AB, fig. 85, be any line, and C any point of it; draw any parallel lines HD, KG, and LF, through A, C, and B respectively; make $CK = CB$, $CG = AC$; and draw the lines HL and GF parallel to AB through K and G respectively.

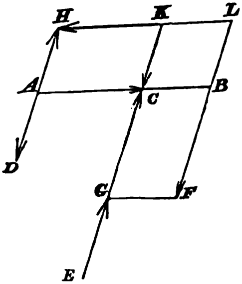
Suppose this figure to be rigid, and that it is acted on by the equal forces represented by KH, AC, GC, and BF; then these forces balance each other, by Proposition VI., because they are equal forces, and act along the sides of the parallelogram $CBLK$, which are all equal; for AC and GC act at C along the sides CB and CK, and we may suppose KH and BF to act at L along the sides LK and LB.

Furthermore, if we produce CG to E, making $GE = CK$, and HA to D, making $AD = AH$; and if we apply, in addition to the former forces, the two pair of equal and opposite forces represented by AH and AD, KC and EG, as is shown in fig. 86; the equilibrium will not thereby be disturbed.

But, by Proposition VI., the forces AH, AC, KH, KC, balance each other, and we shall there-

fore suppose them to be removed: also EG and GC make one force, represented by EC. We have then remaining only the forces AD, EC, and BF, as shown in fig. 84. These forces, therefore, balance each other.

Fig. 86.



Now, by our construction, $AD = AH = KC = CB$, $BF = GC = AC$, and $EC = EG + GC = KC + GC = AC + CB = AB$: hence, the three balancing forces AD, BF, and EC, are represented in magnitude by the lines BC, AC, and AB respectively; or, what is the same thing, the forces AD, BF, and EC, are proportional to the lines BC, AC, and AB. *Which was to be proved.*

Corollary.—If P and Q denote the forces AD and BF, and R the force EC, the conditions of equilibrium may evidently be stated thus, viz.

$$P : Q :: BC : AC,$$

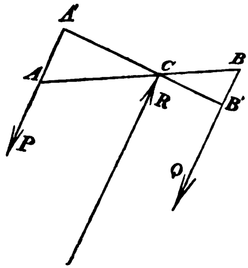
$$\text{and } P + Q = R.$$

Proof of this Proposition deduced from the Principle of the Lever.

By Proposition I. it appears that, if $A'B'$ be a lever, the fulcrum being C (fig. 87), and if the forces P and Q act perpendicularly to $A'B'$ at the points A' and B', the lever will be kept at rest when $P : Q :: B'C : A'C$. Also, by the Corollary to the same Proposition, it appears that the *pressure* on the fulcrum is a force $P + Q$ acting at

right angles to $A'B'$; and consequently, the reaction of the fulcrum is an equal and opposite force.

Fig. 87.



Now, the lever is acted upon, and kept at rest, by the three forces P, Q, and the reaction of the fulcrum, which call R; and these three are parallel forces, such that $P : Q :: B'C : A'C$, and $P+Q=R$. Also, if we draw any line AB through C, meeting the directions of P and Q at A and B, we have, by similar triangles, $B'C : A'C :: BC : AC$, and $\therefore P : Q :: BC : AC$.

Hence it appears, that, if P, Q, and R be parallel forces acting on a rod AB (for we may suppose P and Q to act at A and B), the conditions of their equilibrium are—

$$P : Q :: BC : AC,$$

$$\text{and } P+Q=R.$$

Which was to be proved.

Axiom XIII.

If several forces keep a body at rest, any one of them must be equal and opposite to the resultant of all the rest.

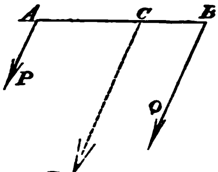
If P, Q, R, S, &c. be the forces, P balances Q, R, S, &c., and therefore P destroys the joint effect of Q, R, S, &c.: consequently, P must be equal and opposite to the resultant of Q, R, S, &c.

Proposition XV.

To find the resultant of two parallel forces acting in the same direction on a rod or rigid body.

Referring to the preceding proposition and fig. 84, it appears by Axiom XIII. that, since P, Q, and R balance each other, the resultant of P and Q must be a force equal and opposite to R. Now, it is shown in the proposition that $R = P + Q$, and $BC : AC :: P : Q$; hence, the resultant of the two parallel forces P and Q, acting in the same direction at the points A and B of a rigid body, is found as follows:—

Fig. 88.



Find the point C which divides the line AB, in the *inverse proportion* of P and Q, that is, which makes $BC : AC :: P : Q$; then the resultant of P and Q is a force $P+Q$ acting at C, parallel to, and in the same direction as, P and Q; as is shown in fig. 88.

Corollary 1.—If $P = Q$, the resultant of P and Q acts at the middle point of AB.

Corollary 2.—*To resolve a force acting at C, fig.* 88, *into two parallel forces, one of which shall act at A, and the other at B.*

This amounts to finding P and Q, when R, AC, and BC are given. To do this, divide AB into as many equal parts or units as there are in R, so as to make AB represent R in magnitude; then the number of these equal parts or units contained in AC will represent Q, and the number in BC will represent P. Thus, let $AC = 20$, $BC = 15$,

and $R = 7$; then $AB = 35$; which, being divided into 7 equal parts each of which contains 5, will represent R. Now, $A\,C$, which is 20, contains 4 of these equal parts, and $B\,C$, which is 15, contains 3; wherefore, P is 3, and Q is 4. In other words, the force 7 acting at C, is resolved into two parallel forces, one equal to 4 acting at B, and the other equal to 3 acting at A.

To do this mathematically, we have,

$$P : Q :: BC : A\,C;$$

$$\therefore P : P + Q \text{ (or } R) :: B\,C : B\,C + A\,C \text{ (or } A\,B);$$

$$\therefore P = \frac{B\,C}{A\,B} R.$$

And in like manner we may show, that

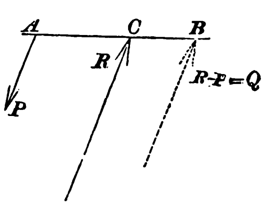
$$Q = \frac{A\,C}{A\,B} R.$$

Which formulæ give the values of P and Q.

Proposition XVI.

To find the resultant of two parallel forces acting on a rigid body in opposite directions.

Fig. 89.



Referring to Proposition XIV. and fig. 84, it appears by Axiom XIII. that the resultant of P and R is a force equal and opposite to Q. Now, since P, Q, and R are represented in magnitude by the lines $B\,C$, $A\,C$, and $A\,B$, respectively, it follows that $Q = R - P$, and $BC : AB :: P : R$;

hence, the resultant of the two parallel forces P and R, acting in opposite directions at the points A and C, is found as follows:—

Produce the line $A\,C$ to B so far that the line may be, what is called, *divided externally* by B in the *inverse proportion* of P to R; that is, find that point B which makes $BC : AB :: P : R$; then the resultant of P and R is a force $R - P$ acting at B parallel to P and R, and in the same direction as R, as is shown in fig. 89. *Which was to be done.*

Observe here that R is the greater of the two forces P and R, and that the resultant is on the same side, and in the same direction, as R. The line $A\,C$ must be produced, not at the extremity A, but at C, that is, at the extremity where the greater of the two forces acts.

Corollary 1.—If $P = R$, no resultant can be found, or, in other words, the two forces do not jointly produce the same effect as any single force whatever.

For, if the two forces P and R were equal, the lines $B\,C$ and $A\,B$, which represent them in magnitude, would also be equal; which is absurd, inasmuch as $A\,B$ always exceeds $B\,C$. Hence, when $P = R$, the reasoning in the proposition is inconclusive, and the rule deduced is of course inapplicable. The fact is, two equal parallel forces acting in opposite directions cannot be balanced by a third force, as is evident from Proposition XIV.; for neither AD and $E\,C$, nor $B\,F$ and $E\,C$ could possibly be equal: and if two such forces cannot be balanced by a third force, they *do not* jointly produce the same effect as any *single force*.

vo equal parallel forces acting in opposite tions are called a *Couple.* There are various ies relating to couples, which form what is l the *Theory of Couples,* and which are of ; importance in the higher parts of Me- ics.

rollary 2.—*To resolve a force acting at B, fig. ito two parallel forces, one of which shall act at id the other at C.*

iis amounts to finding P and R, when Q, and CB are given. To do this, divide AC as many equal parts or units as there are in as to make AC represent Q in magnitude; the number of these equal parts or units iined in AB will represent R, and the ier in BC will represent P. Thus, let AC), $BC = 15$, $Q = 4$; then AC, which is 20, divided into 4 parts each of which is 5, will sent Q. Now AB, which is 35, contains 7 iese equal parts, and BC, which is 15, con- 3; wherefore, R is 7, and P is 3. In other s, the force 4 acting at B is resolved into two lel forces, one equal to 7 acting at C, the equal to 3 acting at A.

B. In this case, observe that the two compo- s, R and P, into which Q is resolved, act in *ite directions,* and that the *greater* of the two onents, namely R, acts the *same way* as Q. athematically, we have, as before,

$$R = \frac{AB}{AC} Q, \text{ and } P = \frac{BC}{AC} Q.$$

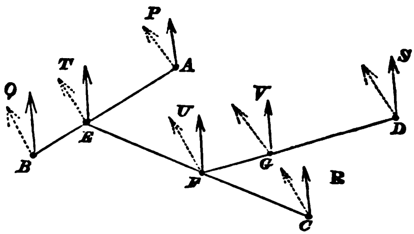
h formulæ give the values of P and R.

Proposition XVII.

To find the resultant of any number of parallel forces acting upon a rigid body.

Let *P, Q, R,* and *S,* fig. 90, be the parallel forces, acting at the points *A, B, C,* and *D* of a rigid body; it is required to find their resultant.

Fig. 90.



Find *T,* the resultant of *P* and *Q,* by Proposition XV., and substitute *T* for *P* and *Q;* then find *U,* the resultant of *T* and *R,* and substitute *U* for *T* and *R;* again find *V,* the resultant of *U* and *S:* then *V* is evidently the resultant of all the forces *P, Q, R, S,* as required. The point of application of *T* will be some point *E* on the line drawn through *A* and *B;* the point of application of *U* will be some point *F* on the line drawn through *E* and *C;* and the point of application of *V* will be some point *G* on the line drawn through *F* and *D.* These points (*E, F,* and *G*) are to be determined by the rule given in the preceding propositions, *i.e.* *B E* and *A E* are to be taken in the same proportion as *P* and *Q, CF* and *EF* as *T* and *R, D G* and *FG* as *U* and *S.*

Corollary 1.—If the forces P, Q, R, S act in the

same direction, *their resultant* V will be equal to *their sum;* for then T will be the sum of P and Q, U the sum of T and R, V the sum of U and S, and therefore V the sum of P, Q, R, and S.

Corollary 2.—If the forces P, Q, R, S be made to act in the directions represented by the dotted arrows (being still parallel to each other), the forces T, U, and V will also be similarly altered *in direction*, but *the points of application* E, F, and G, *will not be altered*, as is manifest by referring to the *rule* for finding the resultant of the parallel forces.

Hence it appears, that if we alter the *directions* of a set of parallel forces (P, Q, R, S), keeping them still parallel to each other, *we do not thereby alter the position of the point of application* (G) *of their resultant.* In other words, if we suppose the forces to turn round their respective points of application (A, B, C, D), still keeping parallel to each other, their resultant will turn round its point of application (G), always, of course, keeping parallel to the forces.

Definition of the Centre of Parallel Forces.—When a set of parallel forces act at certain definite points, (A, B, C, D, suppose,) of a rigid body, their resultant will also act at a definite point, G, the position of which depends solely upon the positions of the points A, B, C, D, and upon the *magnitudes* of the parallel forces, but *not upon the direction* in which they act; so that if the forces be made to act in a different direction, the position of the point G will not be altered thereby, provided no change has been made in the points of application, A, B, C, D, of the forces, or in their *magnitudes.*

The point G is, for this reason, and under these circumstances, called the *Centre of the Parallel Forces.*

Definition of the Centre of Gravity of a Body.—The weight of a body is not really a single force, but is the aggregate or resultant of a set of parallel forces; for the force of gravity, that is, the attraction of the earth, is exercised on every particle of the body: thus every particle is pulled vertically downwards by its own weight. The body therefore is acted upon by a set of parallel forces, namely, the weights of its different particles, and the resultant of these forces is the total weight of the body.

Now, by what has just been proved, these parallel forces have a *centre* where their resultant always acts; *that centre* is called the *Centre of Gravity* of the body.

By holding the body in different positions with respect to the horizon, we change the positions of the different particles with respect to the direction in which the force of gravity acts; or, what is the same thing, we change the direction of the force of gravity with reference to the positions of the different particles. But we do not thus change the weight of the particles, that is, we do not change the magnitudes of the parallel forces; nor do we in any way alter the relative positions of the particles, that is, we do not alter the relative positions of the points where the parallel forces act. Consequently, the position of the centre of the parallel forces, that is, the position of centre of gravity in the body, is not altered by *holding* the body in different positions, or turning *it round* in any way. For instance, if the centre

gravity of a body in the shape of a parallelo-am $ABCD$ be at the intersection of the diago-ls AC and BD (as it may be proved to be) when ; parallelogram is held in the position shown in . 91, it will also be at the intersection of the

Fig. 91.

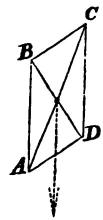
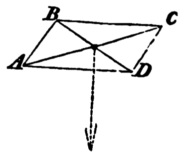


Fig. 92.



gonals when the parallelogram is held as in . 92, or in any other position.

Proposition XVIII.

The weight of a body is a force proportional to the antity of matter in the body, and we may suppose to act at a certain invariable point of the body, mely, the centre of gravity.

By the term *weight* we mean, strictly speaking, e *set of parallel forces* which the attraction of the rth exerts *upon the different particles* of a body. hese forces all act in the same direction, namely, rtically downwards; therefore, as we have shown ove, the resultant of these forces is equal to e *sum of them.* Now the quantity of matter a body depends upon the number and nature its particles. If the substance of which the dy is composed be of the same nature and ality throughout, as we of course suppose it to

be, the particles will all be of equal weight. Consequently, the sum of the forces exerted on the different particles by the attraction of the earth, will be proportional to the number of the particles, that is, to the quantity of matter in the body. The resultant, then, of all these forces, is proportional to the quantity of matter in the body.

It appears, then, that the weight of a body, meaning thereby the set of parallel forces exerted by gravity on its different particles, is equivalent to a single force or resultant which is proportional to the quantity of matter in the body; and since this resultant acts, as we have shown, at that invariable point of the body called the centre of gravity, the truth of the proposition is manifest. *Which was to be proved.*

Corollary.—If the centre of gravity of a body is supported, the body remains at rest. For then the weight, which we may consider to be a force acting at the centre of gravity, is destroyed by the reaction of the support.

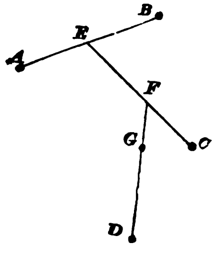
Proposition XIX.

To find the centre of gravity of a number of given bodies, supposed to be rigidly connected with each other, the centre of gravity of each body being known.

Let A, B, C, D, fig. 93, be the known centres of gravity of the bodies, at which points respectively we may suppose their weights act. Then, drawing the line AB, divide it at E, so that the proportion of AE to EB may be the same as that of the

weight B to the weight A: E will be the point of application of the resultant of these two weights (see Prop. XV.), and we may suppose the two weights to act at E. Again, drawing the line EC, divide it, so that the proportion of EF to FC may be the same as that of weight C to weights A and B together: then the three weights, A, B, and C, may be supposed to act at F. Lastly, drawing DF, divide it, so that the proportion of FG to GD may be the same as that of weight D to weights A, B, and C together; then the four weights may be supposed to act at G, and therefore G is the centre of gravity required.

Fig. 93.

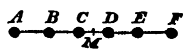


Proposition XX.

The centre of gravity of a set of equally heavy and equidistant particles, rigidly connected together, and forming a straight line, is the middle point of that line.

Let A, B, C, D, E, F, fig. 94, be the particles, and M the middle point of the line AF. Then, because the particles are equidistant from each other, M is half-way between A and F, between B and E, and between C and D; therefore, since the particles A and F are equally heavy, their weights produce the effect of a force acting at M. (Prop. XV.

Fig. 94.



Cor.) In like manner the weights of B and E produced the effect of a force at M; and the same may be said of C and D. Therefore, the weights of all the particles produce jointly the effect of a force acting at M, and therefore M is the centre of gravity of the particles. *Which was to be proved.*

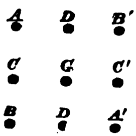
Corollary 1.—Hence the centre of gravity of a rod or beam of uniform weight and thickness throughout, is its middle point. For such a rod may be supposed to consist of a set of equally heavy and equidistant particles forming a straight line.

Corollary 2.—Hence we may always suppose that the weight of a uniform straight rod or line of particles is, so to speak, *collected at its middle point.*

Corollary 3.—By reasoning as above, we may show that the centre of gravity of a flat body in the shape of a circle is the centre of the circle; for we may conceive the whole body to be composed of pairs of equal particles, and each pair to have the centre of the circle midway between them. If the body were in the form of a square, we might in the same way prove that its centre of gravity would be its middle point. In like manner we might show that the centre of a uniform sphere is its centre of gravity; and, in fact, the centre of every body, *which has a centre* properly so called, must be also its centre of gravity.

When we speak of a *body having a centre*, we mean, that its particles are so arranged, that the whole may be supposed to consist of pairs of particles, each pair having the same point midway between them, which point is called the *centre*.

Fig. 95.



Thus, in fig. 95, if the body be composed of the particles AA', BB', CC', DD', and if these be so distributed that the point G is half-way between and in the line joining A and A', also half-way between and in the line joining B and B', also similarly situated with respect to C and C', and with respect to D and D'; then the body is said to have a centre, namely, G.

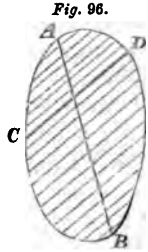
It is clear from this, that the centre of a body which has a centre must be its centre of gravity.

Proposition XXI.

If $ACBD$, fig. 96, *be a flat body or thin board of uniform weight and thinness throughout; and if it be of such a shape with reference to two lines AB and CD, that CD, and all the lines that can be drawn on the board parallel to CD, are bisected by AB; the centre of gravity of the board lies somewhere in the line AB.*

For we may conceive the board to be composed of uniform lines of particles parallel to CD; and if CD, and all the lines that can be drawn on the board parallel to CD, are bisected by AB, the centre of gravity of every one of these lines is at the point where it is intersected by AB. Now we may suppose the weight of each rod to be collected into and act at its centre of gravity: therefore, the weight of the whole board is equivalent to a series of weights acting at different points along the line

Fig. 96.

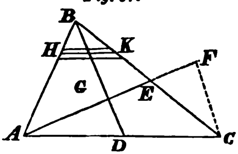


AB; wherefore it is obvious, that the resultant of these weights, and therefore of the whole weight of the board, must act at some point of the line AB. In other words, the centre of gravity must be somewhere in the line AB. *Which was to be proved.*

Corollary 1.—*To find the centre of gravity of a triangle, that is, of a flat body or board in the shape of a triangle.*

Let ABC, fig. 97, be the triangle; bisect AC in D, and join B and D. Then it may be easily shown,* that every line drawn in the triangle parallel to AC, as for instance HK, is bisected by BD. Wherefore, the centre of gravity of the triangle lies somewhere in BD.

Fig. 97.



In like manner, if we bisect BC at E, and join A and E, we may show that the centre of gravity lies somewhere in AE. Consequently, the point G, when BD and AE intersect, must be the centre of gravity of the triangle.

We may prove by measurement that DG is always *one-third* of DB, EG *one-third* of EA. We may show this by Euclid, Book VI., as follows:—

Draw CF parallel to DB to meet AE produced in F; then, by the second Proposition of Euclid, Book VI., since GD and FC are parallel, we have,

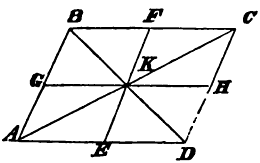
* By Euclid, Book VI., or by measurement. By Euclid we may show that the line BD cuts both AC and HK in the same *proportion*; and therefore, since it bisects AC, it must bisect HK also.

$$FC : DG :: CA : DA :: 2 : 1,$$
and $FC : GB :: CE : EB :: 1 : 1;$

therefore $FC = 2\,DG$, and $GB = FC = 2\,DG$; consequently $DB = DG + GB = 3\,DG$. Q.E.D.

Corollary 2.—If the board be in the shape of a parallelogram $ABCD$, fig. 98, and if we draw EF bisecting AD and BC, and GH bisecting AB and CD, we may show in the same way that the centre of gravity is at K, the point of intersection of EF and GH. K is the point of intersection of AC and BD also.

Fig. 98.



We have now given the rules for the composition of *parallel* forces, and we have dwelt particularly on the case of the system of parallel forces which act upon every body in consequence of the attraction of the earth on the particles of matter. We shall now proceed to investigate the rules for the composition of forces *not parallel*, acting at different points of a rigid body.

COMPOSITION AND EQUILIBRIUM OF FORCES NOT PARALLEL.*

It will be necessary here to employ the principle we have stated in Proposition V. relative to the effect of a set of forces acting on a lever, but as we have not given a rigorous proof of that principle, we must do so now. The student will remember the meaning of the terms employed in the chapter on the lever, namely, *arm*, *moment*,

* *N.B. In all that follows*, the forces are supposed to act in *the same plane.*

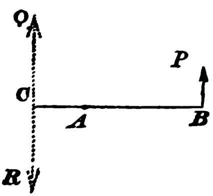
like and unlike moments: the *arm* of a force is the perpendicular upon its direction from the fulcrum; the *moment* of a force is the product of the force multiplied by its arm; forces which tend to turn the lever the same way round the fulcrum are said to have *like* moments, and those that tend opposite ways, *unlike*.

Proposition XXII.

If P be a force acting on a lever, the moment of P expresses the magnitude of that force which, acting at an arm unity, produces the same effect as P in tending to turn the lever.

Let A, fig. 99, be the fulcrum, and $B\,A$ the arm of P, produce $B\,A$ to C, making $A\,C$ equal to unity; suppose for a moment, that AB contains 5 units, and apply (as we may) two opposite forces, Q and R, each of which is parallel to P, and equal to 5 times P.

Fig. 99.



Now, since Q is 5 times P, and $A\,B$ 5 times $A\,C$, the resultant of P and Q is a force acting at A (Prop. XV.); but this force cannot produce motion, since A is a fixed point, and consequently P and Q balance each other; we may therefore remove P and Q, and then there remains only R, which is a force acting at an arm unity, and equal to 5 P. In like manner, if AB were 10, we should have $R = 10\,P$, and if $A\,B$ were 15, we should have $R = 15\,P$; and in general, whatever number $A\,B$ is, R will be equal to P multiplied by that number. In other words, R will be equal to $P \times A\,B$, which is the moment of P.

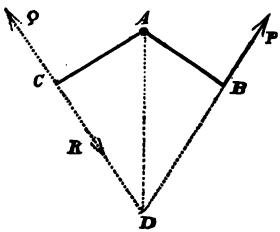
Hence it appears that we may remove P, and apply in place of it a force acting at an arm unity, and equal in magnitude to the moment of P; in other words, the moment of P expresses the magnitude of that force which, applied at an arm unity, produces the same effect as P. *Which was to be proved.*

Proposition XXIII.

Two equal forces acting on a lever at equal arms, and tending to turn it the same way, produce the same effect, and may be substituted one for the other.

Let P, fig. 100, be any force acting on a lever, and $A\,B$ its arm, A being the fulcrum; draw any line $A\,C$ equal to $A\,B$, and at C apply two opposite forces Q and R, each equal to P and perpendicular to $A\,C$; let the directions of these forces be produced to meet in D, and join $D\,A$. Then, since $A\,B$ and $A\,C$ are equal, it is clear that $D\,A$ bisects the angle $B\,D\,C$; therefore, by Axiom VIII., since P and Q are equal, their resultant is a force acting along $D\,A$, which force can produce no motion, A being a fixed point. Hence P and Q balance each other, and may be removed, and then R alone remains.

Fig. 100.

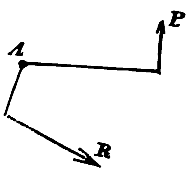


Wherefore it appears that we may remove P and put R in its place; in other words, two equal forces, R and P, acting at equal arms, and tending to turn the lever the same way, produce the same

effect, and we may substitute one for the other. *Which was to be proved.*

Corollary 1.—From this proposition and the preceding it appears, that, if P be any force acting on a lever, we may remove it, and apply in its place a force R, fig. 101, which is equal to the moment of P, and acts *in any direction* on the lever at an arm equal to unity. Of course R must tend to turn the lever the same way as P.

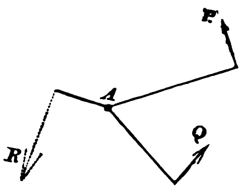
Fig. 101.



Corollary 2.—Hence, two forces whose moments are *equal* and *like*, produce the same effect upon a lever, and may be substituted one for the other.

For, let P and Q, fig. 102, be the two forces, and let R be a force acting at an arm unity, and equal to the moment of P, and therefore to that of Q also. Then, by what has been proved, P produces the same effect as R, and the same is true of Q also: therefore P and Q produce the same effect.

Fig. 102.



Corollary 3.—Hence two forces whose moments are *equal* and *unlike*, balance each other.

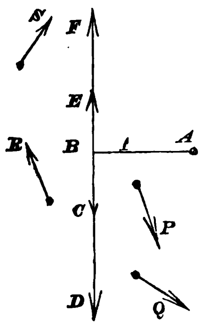
For, in the above figure, suppose that Q is reversed in direction: then, instead of producing the same effect as R, it will produce the same effect as a force equal and opposite to R; therefore, since P produces the same effect as R, it is clear that P and Q will balance each other.

Proposition XXIV.

To estimate the effect of a given set of forces acting on a lever.

Let the forces be $P, Q, R,$ and S, fig. 103, A the fulcrum, and DF any line drawn in the lever at a distance unity from the fulcrum: on this line measure the portions BC, CD, BE, and EF, equal respectively to the moments of the forces $P, Q, R,$ and S. Then, by the first Corollary of the previous Proposition, we may remove $P, Q, R,$ and S, and put in their place the forces represented by BC, CD, BE, and EF. Now taking the sum of the forces BC and CD, and the sum of BE and EF, and subtracting the lesser sum from the greater, the difference will be the resultant of the forces, and it will act in the direction of those composing the greater sum. Thus we have compounded $P, Q, R,$ and S into a single force acting along the line DF.

Fig. 103.



Hence we have the following *Rule* for estimating the effect of a given set of forces on a lever, viz.:

Find the sum of the moments of the forces which tend to turn the lever one way, and the sum of the moments of those which tend the opposite way, and subtract the lesser sum from the greater; then a force equal to the difference, acting at a distance unity from the fulcrum, and tending the same way as the forces whose moments compose the greater sum, will produce the same effect upon the lever as the given set of forces.

Corollary.—If the two sums be equal their difference will be nothing, and the forces will then balance each other. Hence we have the following *Rule* for the equilibrium of forces acting on a lever.

A set of forces acting on a lever will balance each other, when the sum of the moments of the forces tending to turn the lever one way is equal to the sum of the moments of those tending the opposite way.

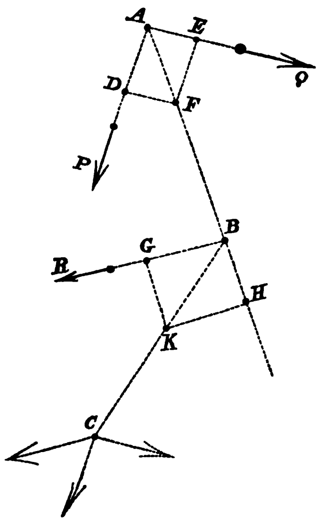
This Rule is often called the *Principle of the Lever*, or the *Principle of the Equality of Moments.*

Proposition XXV.

When a lever is kept at rest by given forces, to find the pressure they exert upon the fulcrum.

Let C be the fulcrum of the lever, and P, Q, and R the given forces acting along the lines $A P$, $A Q$, and $B R$, fig. 104.

Fig. 104.



Suppose (as we may) that P and Q act at A, take the lines $A D$, $A E$ to represent them, and complete the parallelogram $A D F E$; then P and Q are equivalent to the force represented by $A F$. Produce AF to meet the direction of R in B, suppose (as we may) that R and the force $A F$ act at B, take the lines BG and

BH to represent them, and complete the parallelogram $BGKH$; then the forces R and AF are equivalent to the force represented by BK.

Hence the given forces P, Q, and R are together equivalent to the single force BK, and therefore BK must keep the lever at rest. Now a single force acting on a lever, in a direction *not* passing through the fulcrum, cannot keep the lever at rest; hence the direction of the force BK must pass through the fulcrum C, and consequently we may suppose C to be the point of application of this force.

It appears, therefore, that the effect of the forces P, Q, and R, is, to exert on the fulcrum a pressure represented in magnitude and direction by the line BK. *Which was to be determined.*

Corollary.—The force BK, supposed to act at C, may be resolved into two forces respectively parallel and equal to BG and BH, or (what is the same thing) to R and AF; and then, the force parallel and equal to AF may be resolved into two forces respectively parallel and equal to AD and AE, or (what is the same thing) to P and Q. Hence the force BK, supposed to act at C, may be resolved into three forces respectively parallel and equal to the forces P, Q, and R.

Thus the given forces P, Q, and R, are equivalent to the force BK, and BK is equivalent to three forces, which act at the point C, and are respectively parallel and equal to P, Q, and R; in other words, we may suppose that the forces P, Q, and R are removed to the point C, and that they act upon that point, in directions respectively parallel to their original directions AP, AQ, and BR.

Hence, *when a set of forces keep a lever at rest,*

we may suppose them to be removed to the fulcrum, making them to act upon that point, each parallel to its original direction.

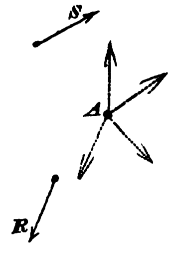
This is a *principle* of great use in various cases.

Proposition XXVI.

To determine under what circumstances a set of forces acting on a perfectly free rigid body, will keep it at rest.

Let P, Q, R, and S, fig. 105, be a set of forces acting on a rigid body and keeping it at rest; then, though the body is perfectly free, we may suppose *any* point of it, say A, to become fixed, (Axiom IV.); in other words, we may suppose the body to become a lever, A being the fulcrum. This being the case, if we find the sum of the moments of the forces which tend to turn the body one way round A, and the sum of those which tend the opposite way, the two sums must be equal, by Cor. Prop. XXIV. Furthermore, by Cor. Prop. XXV. we may remove the forces P, Q, R, and S to the point A, and suppose them to act on that point, each parallel to its proper direction; the forces thus removed must balance *each* other at A, for, if they do not, A, which is *really* a free point, will not remain at rest.

Fig. 105.



Hence the following are the conditions necessary, in order that a set of forces may keep a free body at rest.

1. The forces must satisfy the principle of the lever, with reference to *any* point of the body considered as fulcrum; that is to say, the sum of the moments of the forces, which tend to turn the body one way about that point, must be equal to the sum of those tending the opposite way.

2. The forces must balance each other, on the supposition that they all act at the same point, each parallel to its proper direction.

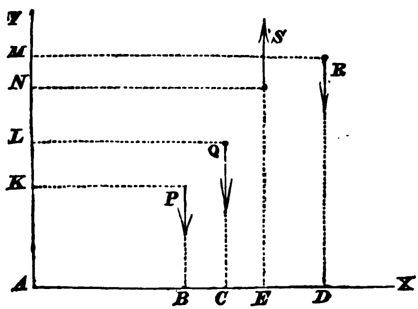
These are called the *conditions of equilibrium* of a free rigid body.

Proposition XXVII.

To determine the centre of gravity of a given set of particles forming a rigid body, by means of the preceding Proposition.

Let P, Q, and R, fig. 106, be the given particles, A any given point, $A X$ a horizontal line drawn through A, and $P B$, $Q C$, and $R D$ vertical lines; let S be the centre of gravity of the particles, and $S E$ a vertical line; then the weights of the particles are forces acting downwards along

Fig. 106.



the lines $P\,B$, $Q\,C$, and $R\,D$, and the result these forces is a force equal to the sum c weights acting downwards along $S\,E$. He force equal to the sum of the weights P, (R, acting upwards along $E\,S$, will balance and R. Therefore, by the preceding Propo the sum of the moments of the three weig being considered as fulcrum, must be equal moment of a force equal to the sum of the w acting upwards along $E\,S$; in other words, multiply the weight of P by $A\,B$, that of $A\,C$, that of R by $A\,D$, and add the pro together, the result must be equal to the s the weights multiplied by $A\,E$: or, in symb

$$P\times A\,B+Q\times A\,C+R\times A\,D=(P+Q+R)\times$$

and therefore,

$$A\,E=\frac{P\times AB+Q\times AC+R\times A\,D}{P+Q+R}.$$

Hence, *if we divide the sum of these produ the sum of the weights, we find* $A\,E$.

Again, let $A\,Y$ be a vertical line, and $Q\,L$, $R\,M$, and $S\,N$ horizontal.

Now we do not alter the position of the c of a set of parallel forces by turning them their points of application through any angle may therefore suppose the forces above consi to act horizontally, as shown by the dotted ar and S will still be their centre. And then we show, just as before, that, *if we divide the s the products (the weights of* P, Q, *and* R *being tiplied by* $A\,K$, $A\,L$, *and* $A\,M$ *respectively) b sum of the weights, we shall find* $A\,N$*: or, in sy*

$$A\,N=\frac{P\times A\,K+Q\times A\,L+R\times A\,M.}{P+Q+R}$$

Having thus determined $A E$ and $A N$, the position of the centre of gravity S is manifestly obtained.

Example.—Suppose that the weights of P, Q, and R are 1, 2, and 3; $A B$, $A C$, and $A D$, 6, 9, and 10; $A K$, $A L$, and $A M$, 3, 6, and 7: then the sum of the weights is 6, and the products of the weights and the first set of distances are 6, 18, and 30, which added give 54; therefore AE is 54 by 6, or $AE = 9$: in like manner the products of the weights and the second set of distances are 3, 12, and 21, which added give 36: therefore $A N$ is 36 by 6, or $AN = 6$.

Hence, if we measure AE equal to 9, and $A N$ equal to 6, and draw a horizontal line through N, and a vertical line through E, the intersection of these lines will be the centre of gravity of the three particles.

Corollary.—The lines $A B$, $A C$, $A D$, and $A E$, are the *distances* (*i. e.* the *perpendicular* distances) of the points P, Q, R, and S, from the line $A Y$: also, though we have supposed $A X$ to be horizontal, and $A Y$ vertical, it is evident that, so far as the above reasoning is concerned, AX and $A Y$ may be *any* two lines at right angles to each other. Hence, we may state the *Rule* for finding the centre of gravity as follows:—

Multiply the weight of each particle by its distance from any given line $A Y$, add the products together, and divide the result by the sum of the weights; then the quotient thus obtained is the distance of the centre of gravity of the particles from the line $A Y$.

In this way we may find the distance of the centre of gravity from any two given lines, and so determine its position.

EXAMPLES.

Examples of finding the Resultant of Parallel Forces.

N.B. In these examples the *point of application* of the resultant is always to be found.

Ex. 1.—Find the resultant of the parallel forces 5 and 7 acting in the same direction at the extremities of a rod whose length is 24.

Ex. 2.—On the same supposition, except that the forces are 4 and 12; find the resultant.

Ex. 3.—On the same supposition, except that the forces are 6 and 18, acting in opposite directions; find the resultant.

Ex. 4.—On the same supposition, except that the forces are 7 and 11, acting in opposite directions; find the resultant.

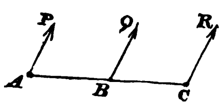
Ex. 5.—Resolve the force 20 into two parallel forces, one of which shall act at a distance 4, and the other at a distance 1 from the force 20.

Ex. 6.—A force 48 acts at a point C of a rod AB, such that $AC = \frac{5}{19} CB$; resolve 48 into two parallel forces, which shall act at the extremities of the rod.

Ex. 7.—On the same supposition, only that C is unknown; find C, when the difference between the two parallel forces into which the force C is resolved is 8.

Ex. 8.—The force 48 acts at A, resolve it into two parallel forces which shall act at B and C, supposing that $AB = \frac{12}{7}$ of AC.

Fig. 107.



Ex. 9.—If the parallel forces P, Q, and R, act at the points A, B, and C respectively, fig. 107; find their resultant, when $P = 4$, $Q = 6$, $R = 8$, $AB = 20$, $BC = 36$.

Ex. 10.—If $P = 2Q$, and $P + Q = R$, and $AC = 4 = 2BC$; find the point where the resultant acts.

Ex. 11.—If $P = 3$, $Q = 5$, $R = 10$, and $AB = 8$; find BC, so that the resultant may act at a distance 10 from A.

Ex. 12.—If $P = 4$, $Q = 2$, $R = 3$, $AB = 12$, $BC = 16$; and if R act in the opposite direction to P and Q; find the resultant.

Ex. 13.—If the 6 parallel forces, 1, 2, 3, 4, 5, 6, act in the same direction, at equal distances of 1 foot from each other in order; find how far from the first of the forces the resultant acts.

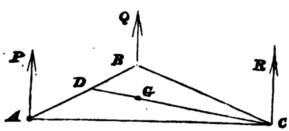
Ex. 14.—Determine the same when the forces 2, 4, and 6, act in the opposite direction to the others.

Ex. 15.—Determine the same when the forces 1 and 6 act in the opposite direction to the rest.

Ex. 16.—On the same supposition as in Ex. 12, except that $P = 1$, find the resultant, and explain the case.

Problem XIV.

Fig. 108.



ABC, fig. 108, is a triangular board, held up by three strings, AP, BQ, CR; to find what tension the weight of the triangle produces on each string; and, if any additional weight be placed at any point on the triangle, to find how much the tension on each string is increased thereby.

Let CD be drawn to D, the middle point of AB, and make $DG = \frac{1}{3} DC$, in which case G will be the centre of gravity of the triangle, and the weight of the triangle, which we shall denote by W, will be a force acting vertically downwards at G.

Resolve this force into two parallel forces, one to act at D, and the other at C, which will be respectively $\frac{2}{3} W$ at D, and $\frac{1}{3} W$ at C, because $GC = 2\,GD$. Again, resolve $\frac{2}{3} W$ acting at D into two parallel forces, one to act at A and the other at B, which will be $\frac{1}{3} W$ at A. and $\frac{1}{3} W$ at B. Thus the weight W is resolved into three parallel forces, acting at A, B, and C, and each equal to $\frac{1}{3} W$. Whence it follows that one-third of the weight is thrown on each string, and therefore the tension on each string is $\frac{1}{3} W$.

Again, suppose that G' is some other point of the triangle, and that a weight W' acts at this point: then, drawing CD' through G', resolve W' acting at G' into two parallel forces, one to act at D', and the other at C; and resolve the force at D' into two other parallel forces, one to act at A, the other at B. Thus W' will be resolved into three parallel forces, which act at A, B, and C, and which are therefore the tensions upon the three strings produced by W'.

If we add $\frac{1}{3} W$ to each of these three tensions, we shall find the total tension on each string, arising from the two weights W and W'. *Which was to be done.*

Ex. 1.—If $W' = W = 30$lbs., and G' coincide with D; find the tension on each of the strings.

Ex. 2.—If $2\,W' = W = 24$, and G' is on the line AB, AG' being double BG'; find the tensions on each of the strings.

Ex. 3.—If G' be midway between D and C, and if $W=9$, $W'=8$; find the tension on each of the strings.

Ex. 4.—If $AD = 2\,D'B$, $D'G' = 2\,G'C$, $W=9$, and $W'=27$; find the tensions on each of the strings.

Ex. 5.—If $3\,AD'=5\,D'B$, $4\,D'G'=5\,G'C$, $W=9$, and $W'=9$; find the tension on each of the strings.

Ex. 6.—If $W=9$, and the tensions on the strings, AP, BQ, CR, are respectively 4, 4, and 7; find the weight of W', and the position of D' on AB, and of G' on $D'C$.

Ex. 7.—If $W=27$, and the tensions are respectively 12, 14, and 17; find the position of D' on AB, and of G' on $D'C$.

Ex. 8.—If $W=15$, $W'=10$, tension on $AP=17$, and tension on $BQ=17$; find tension on CR, and the position of G'.

Ex. 9.—If the weight of the triangle be 12, and each string can bear a tension of 8; find what weight W' may be put at G' without breaking any string, supposing that D' is the middle point of AB, G' the middle point of $D'C$.

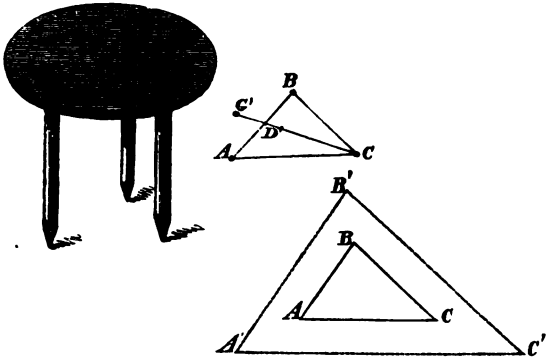
Ex. 10.—On the same supposition, except that $W'=9$, and the position of G' on $D'C$ is not given; find the extreme positions where W' may be placed without breaking any string.

Ex. 11.—On the same supposition, except that D' as well as G' is not given in position; find the extreme position of D' on AB, and of G' on $D'C$, so that no string be broken.

Ex. 12.—A table is supported on three legs, A, B, C, fig. 109; show how to determine how much of *the weight* each leg bears, the position

of the centre of gravity, G, of the table being supposed to be known with reference to the legs.

Fig. 109.



Ex. 13.—If $AB = 3$, $BC = 4$, $AC = 5$, and if G be equidistant from the three legs; find how much weight is thrown on each.

In this case, a triangle whose sides are 3, 4, and 5, must be constructed; and then, to find where G is, we must draw a perpendicular to one side, say the side 3, from the middle point of the side 3; and from the middle point of another side, say the side 4, we must draw a perpendicular to the side 4. It will be found, and may be proved, that the point where these perpendiculars meet is equidistant from the three angular points, and is therefore the point G required. Having found G, we must proceed as above to find how much weight is thrown on each leg.

PROBLEM XV.

If a weight be placed on the table outside the limits of the triangle $A\,B\,C$, fig. 109 *; to find whether the table will upset or not.*

Let G' be the point where the weight is placed, draw $C\,G'$ meeting $A\,B$ at D'; also, let W' denote the weight. Then we must resolve W' acting at G' into two parallel forces, one to act at D', and the other at C. The force at D' will act the same way as W', that is, downwards; the force at C will act the opposite way, that is, upwards. The force at D' may be resolved into two others, one at A and one at B, both downward forces. We must also, as above, find how much of the weight of the table is thrown on each leg.

Then it is clear that the effect of W' on the legs A and B, is to press them against the ground, but the effect on the leg C is upwards, and tends to raise it from the ground. If, therefore, the upward force at C caused by W', be greater than the downward force at C caused by the weight of the table, C will rise, and the table will be upset; otherwise the table will not be upset.

Ex. 1.—Supposing that the centres of gravity of the triangle and table coincide, and that the weight of the table is 90 lbs.; determine how far a weight of 30 lbs. may be placed on the table outside the limits of the triangle, without upsetting the table.

In this example, one-third of the weight of the table is thrown on each leg; therefore 30 lbs. acts at C downwards, and consequently the upward effect produced by W' on C must not exceed 30 lbs., or the table will upset. At most, then,

the upward effect produced by W' at C is 30lbs.; wherefore W' being 30 lbs., $G'D'$ must be equal to $D'C$; for by Prop. XVI. Cor. 2,

$$G'D' : D'C :: \text{component of } W' \text{ acting at } C : W' :: 30 : 30.$$

Wherefore $G'D'$ is equal to $D'C$.

This, then, is the limit which determines how far W' may be placed outside the triangle, beyond AB, without upsetting the table; namely, $G'C$ must never exceed $2D'C$. Similar reasoning will apply to the other sides of the triangle, and therefore the condition of not upsetting is this:—If we draw from G' a line to any angle of the triangle, cutting the side opposite that angle at D', the distance of G' from the angle must not exceed double the distance of D'.

If we form the triangle $A'B'C'$ by drawing lines parallel to the sides of ABC, $A'B'$ being twice as far from C as AB, $B'C'$ being twice as far from A as BC, and $C'A'$ being twice as far from B as CA; it is easy to see that the weight W may be placed anywhere inside the triangle $A'B'C'$ consistently with the condition just stated; and therefore $A'B'C'$ shows the limits outside which W' must not be placed.

Ex. 2.—If the centre of gravity of the table be at the middle point of BC; find how far beyond AB a weight four times that of the table may be placed without upsetting it.

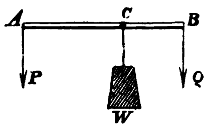
Ex. 3.—On the same supposition as in Example 1, except that $W' = 60$ lbs.; find the limits beyond which W' must not be placed, in order that the *table* may not upset.

Problem XVI.

If AB, fig. 110, be a lever or bar, by which a weight suspended from the point C of it is supported on the shoulders of two men; to find how much of the weight is thrown on each man. To find also how much of the weight is thrown on each man when there are three supporting it, as is shown in fig. 112.

Let P and Q be the pressure which is exerted at A and B on the men's shoulders, by the bar AB; these pressures must be together equivalent to W, in other words, W must be equivalent to the resultant of P and Q. Wherefore, since we may take the lines AC, BC, and AB to represent the forces Q, P, and W respectively in magnitude, we have,

Fig. 110.



$$P : W :: BC : AB,$$
$$\text{and } Q : W :: AC : AB;$$

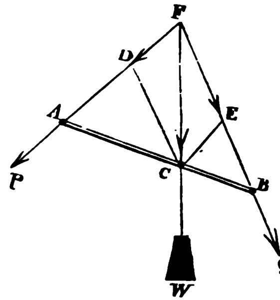
which proportions give P and Q when W, AC, and BC are known.

We have here assumed, however, that P and W act vertically downwards; but this is not necessarily the case, for the men might so hold the lever on their shoulders as to press obliquely as well as vertically upon it, and of course P and Q must be equal and opposite to whatever forces the men exert in supporting the lever. We must therefore consider the problem more generally, by supposing that P and Q are not vertical forces. We shall also, at the same time, suppose that the bar is *inclined to the horizon*, as it would be if

the men were going up a hill, or if one man was taller than the other.

Let AF and BF be the directions of the forces which the men exert in supporting the bar AB, fig. 111, then the direction of W must pass through the point F where these directions meet, by II. p. 90 : take FC to represent W, and draw CD and CE parallel to the lines BF and AF. Then the force FC may be resolved into the two forces FD and FE, and these two forces are the pressures exerted on the men's shoulders. We have therefore,

Fig. 111.



$$P : W :: FD : FC,$$
$$Q : W :: FE : FC.$$

From these proportions we may find P and Q when W is given, and FD, FC, and FE determined by measurement or calculation, or otherwise.

Ex. 1.—The men press vertically, W is 100lbs., and AC is three times BC; find the pressures on the men's shoulders.

Ex. 2.—The hinder man B presses forward as well as upward, so that the force he exerts makes an angle of 30° with the vertical; the bar is horizontal, $AC = CB$, and $W = 100$lbs.; find the pressure on each man's shoulder.

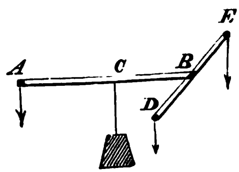
Ex. 3.—The bar is inclined at an angle of 30° *to the horizon*, and everything else is the same as

in Example 2; find the pressure on each man's shoulder.

In Examples 2 and 3 draw the bar and the lines BF and CF first, then join A and F.

If the bar, supposed to be like the letter T, *be supported by three men,* A, D, and E, as is shown in fig. 112, we must first resolve W into two forces, one acting at A, the other at B; we must then resolve the force at B into two others, one at D, and the other at E. We shall thus distribute W on the three points of support.

Fig. 112.



Supposing the men to press vertically, in this case we have,

$$\text{force at } A : W :: BC : AB;$$
$$\text{force at } B : W :: AC : AB;$$

$$\text{and therefore force at } B = \frac{W \times AC}{AB};$$

$$\text{then, force at } D : \frac{W \times AC}{AB} :: BE : DE,$$

$$\text{and force at } E : \frac{W \times AC}{AB} :: BD : DE.$$

By these proportions the pressures at A, D, and E are determined.

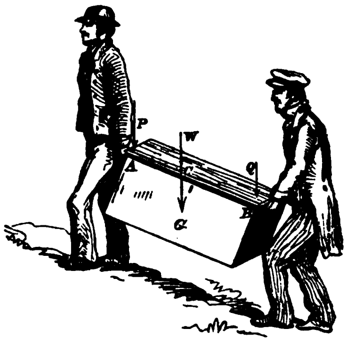
Ex. 1.—W=100lbs., $AC = CB$, and $DB = BE$; find the three pressures.

Ex. 2.—What must be the proportion of AC to CB, and of DB to EB, so that the load may be equally distributed on the three shoulders?

Problem XVII.

Two men carry a large box up a hill, as is represented in fig. 113; *to find how much of the weight is thrown on each.*

Fig. 113.



Suppose that the men hold the box at A and B, and exert vertical forces in supporting it. Let G be the centre of gravity of the box, and draw $G\,C$ vertically to meet $A\,B$ at C. Then, if W be the weight of the box, P and Q the pressures at A and B exerted by W, we have, as in the preceding problem,

$$P : W :: B\,C : A\,B;$$
$$Q : W :: A\,C : A\,B;$$

whence P and Q may be determined.

It is clear that, if G be at the middle point of the box, $B\,C$ is greater than $A\,C$, and therefore the hinder man has less work to do than the other.

Ex. 1.—$A\,B$ is 4 feet, G is 1 foot perpendicularly

e middle point of AB, the inclination of
the horizon is 45°, and W is 100 lbs.; find
ch more the first man has to support than
r.

.—On the same supposition, except that
ler man exerts a forward as well as a ver-
ssure, so that the whole force he exerts is
at 30° to the vertical; find P and Q.

is case draw a line from B at 30° to the
to meet $G\,C$ produced, at F suppose; join
A, and draw from C lines parallel to $B\,F$
, so forming a parallelogram. Then, if we
C to represent W, the sides of this paral-
will respectively represent P and Q.

CHAPTER V.

PROBLEMS RELATING TO THE CENTRE OF GRAVITY.

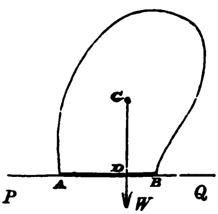
PROBLEM XVIII.

If a body be placed on a horizontal or inclined plane, to determine under what circumstances it will stand without upsetting.

Let $A\,C\,B$ be the body, which we shall suppose to have a portion, $A\,B$, of its surface flat, and let $P\,Q$ be a horizontal plane, on which $A\,B$ is placed; let G be the centre of gravity of the body, and draw a vertical line $G\,D$ through G, to meet the horizontal plane at D.

In the first instance, suppose that the point D, where the vertical $G\,D$ from the centre of gravity meets the horizontal plane, falls *within the base*, $A\,B$, fig. 114; then the force which presses the body against the horizontal plane, has no tendency to upset it, for the weight W of the body is the force which presses it against the horizontal plane, and we may suppose that this force acts at D. Now, a force acting vertically downwards at D, cannot turn the body over either on the right

Fig. 114.



hand, or on the left; for, if the body turns over on the right hand, it must do so by turning about the point *B;* but the force *W* clearly resists such a motion. Again, if it turns over on the left hand, it must do so by turning about the point *A;* but the force *W* resists a motion of this kind likewise. Consequently, the body cannot turn over either on the right hand or the left hand, as far as the action of the force *W* is concerned. In this case, therefore, the body will stand without upsetting.

In the second place, suppose that the point *D*, where the vertical through the centre of gravity meets the horizontal plane, falls *without the base*, *A B*, as in fig. 115; then *W* has clearly a tendency to make the body turn over on the right hand about the point *B*, and therefore, since there is nothing to prevent such a motion, it will take place. If *D* fell on the other side of *A B*, as is shown in fig. 116, the tendency of *W* would be to make the body turn over on the left hand, about the point *A*, and, there being nothing to prevent such a motion, it would take place.

∴ Hence it appears, that, if the vertical through the centre of gravity meets the horizontal plane at a point *within* the base, the body will stand without upsetting; but if that *point falls without* the base, the body

Fig. 115.

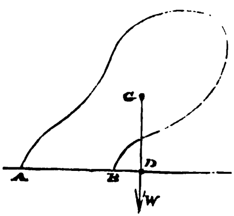
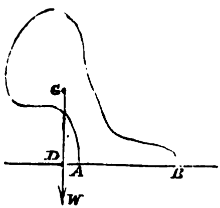


Fig. 116.



will upset, by turning over on that side on which the point falls beyond the base.

Fig. 117.

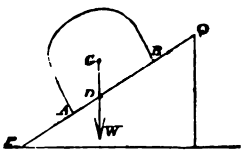


Fig. 118.

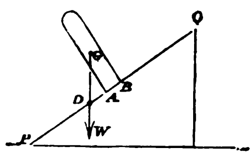
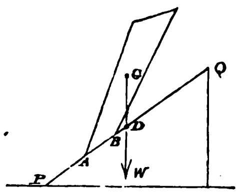


Fig. 119.



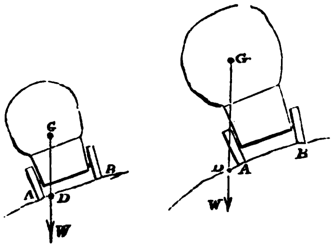
If the *plane*, *P Q*, were *inclined* to the horizon, as in figs. 117, 118, 119, the same reasoning would apply, and the same conclusion follow; namely, the body will remain steady, or upset, according as the point *D*, where the vertical through the centre of gravity meets the plane *P Q*, falls within or without the base. Thus, in fig. 117 the body will remain steady, but in figs. 118 and 119 it will upset; in the former case, by turning over on the left hand about the point *A*, and in the latter case, by turning over on the right hand about the point *B*.

Of course, when the body is thus placed upon an inclined plane, it is supposed that it cannot *slip* down the plane, either from the roughness of the plane, or some other cause; in fact, we assume that the body can only move by turning over at either *A* or *B*.

A familiar illustration of this is the case of a wagon going along a road which inclines to one side, fig. 120. If *G* be the centre of gravity of the wagon, including the load, *A* and *B* the points where the wheels rest on the road, and *G D* the *vertical* through the centre of gravity meeting the

at D; then $A\ B$ may be regarded as the base which the wagon stands, and if D fall within

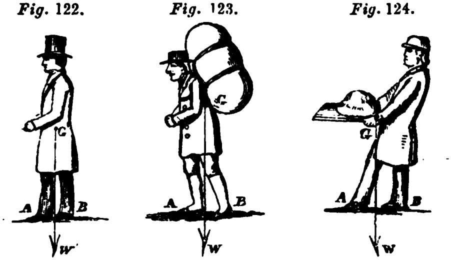
Fig. 120. *Fig.* 121.



', as in fig. 120, the wagon will not upset; if the load be too high, as in fig. 121, so that centre of gravity is in a higher position above ground, the vertical through G will fall outthe base at the point D, and the consequence be that the wagon will upset. Hence the ortance of not putting too high a load on ons is obvious.

. man must stand in a vertical position, in er that the vertical through his centre of gravity fall within the limits of his feet, which form base on which he stands, fig. 122; but if he ies a load, as in figs. 123 and 124, he must

Fig. 122. *Fig.* 123. *Fig.* 124.



incline himself in such a way that the v
through the centre of gravity of the whol
composed of the man and the load togethe
fall within the limits of his feet. This accou
the various attitudes assumed by porters,
and other persons carrying loads.

PRINCIPLE OF THE DESCENDING TENDENCY (
CENTRE OF GRAVITY.

We have on a former occasion stated th
ciple, that the centre of gravity always a
the lowest position it can; or, to speak mo
rectly, the centre of gravity will always m
it can begin to move by descending, but w
move if it cannot do so. If we consider h
centre of gravity is capable of moving
case, we shall find no difficulty in applyi
principle. Thus, in the case represented
125, if we describe arcs of circles, $G\,E$ an

Fig. 125.

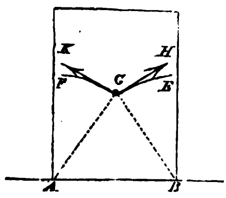
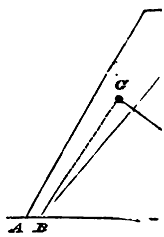


Fig. 126.



from G about B and A respectively as cent
centre of gravity G can move, either by t
about B and describing the circular arc G
by turning about A and describing the c

arc GF. The arrow GH, which is at right angles to BG, shows the direction in which the former motion would commence, if it took place; and the arrow GK, which is perpendicular to AG, shows the direction in which the latter motion would commence, if it took place. Now, both these arrows indicate *an ascending motion*, and therefore, according to the principle stated, the centre of gravity will not move at all.

On the contrary, in the case represented by fig. 126, if a circular arc be described from G about B as centre, the arrow GH at right angles to BG indicates a descending motion, and therefore G may begin to move by descending; consequently, motion will ensue about the point B.

The same considerations will apply to the case where the body is placed on an inclined plane; we must describe circular arcs from G about A and B, to show how G may move, and if G in neither case would begin to move by descending, no motion will take place. The direction in which G would begin to move, say about B, is shown by drawing an arrow from G at right angles to BG, to indicate the direction in which the circular arc runs from G; for, when a point describes a circle it always runs perpendicularly to the line drawn from it to the centre.

We may, then, state the principle of the descending tendency of the centre of gravity in the following manner:—

Draw an arrow to indicate the direction in which the centre of gravity may begin to move; then, if that arrow is inclined downwards, the centre of gravity will *move in that* direction; but if the

arrow is not inclined downwards, that is, if it points horizontally, or is inclined upwards, the centre of gravity will not move in that direction.

A body having a fixed point, about which it may freely turn, is a good instance to which this principle may be applied. Let *A B*, fig. 127, be such a body, *C* the fixed point, or point of suspension as it is called, and *G* the centre of gravity. In the first instance, suppose that *G* is *vertically below* *C*, describe through *G* the circular arc *F G E* about *C* as centre, and draw the arrows *GH* and *GK* at right angles to *CG*. Then the centre of gravity may move so as to describe the circular arc *F G E*, and the arrows *GH* and *G K* show the two directions in either of which that motion must begin. Now, *C G* being vertical, these arrows point horizontally; wherefore, by the principle we are here considering, no motion will take place.

Fig. 127.

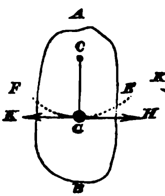
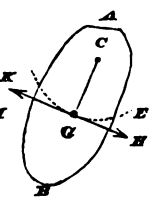
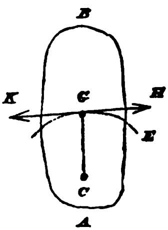


Fig. 128.



In the second place, suppose that *G* is *below* *C*, but that *C G* is *not vertical*, as is represented in fig. 128. There the arrow *G H* is inclined downwards, and therefore the centre of gravity will begin to move in the direction *G H*.

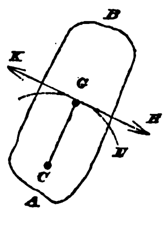
Fig. 129.



Thirdly, let *G* be *vertically above* *C*, as is shown in fig. 129. In this

case both the arrows point horizontally, and therefore no motion will take place.

Fig. 130.



Lastly, let *G* be *above C*, but *not vertically*, as is shown in fig. 130. In this case the arrow *G H* inclines downwards, and therefore the centre of gravity will begin to move in the direction *G H*.

STABLE AND UNSTABLE EQUILIBRIUM.

There is a remarkable difference between the two cases of equilibrium represented by figs. 127 and 129, as will be manifest by referring to figs. 128 and 130. In the case represented in fig. 128, the centre of gravity will move in such a way as to bring the line *C G* back to the vertical position, and therefore the tendency of the body is to fall back into its position of rest. In fact, if we suppose the body to be suspended in the manner represented in fig. 127, and to be slightly disturbed from that position of equilibrium, by being pushed a little on one side, so as to assume the position in fig. 128; then, when the body is left to itself, it will fall back again towards its original position of rest in fig. 127.

On the contrary, in the case represented by fig. 130, the centre of gravity will move in such a way as to turn the line *C G* round away from the vertical position; and therefore the tendency of the body is, not to fall back to a position of rest, but to fall away from it. In other words, if we suppose the body to be placed in the manner represented *in fig. 129,* and to be slightly disturbed

from that position of equilibrium, by being pushed aside a little, so as to assume the position in fig. 130; then, if left to itself, the body will fall away from its original position of rest in fig. 129, and swing quite round.

There is, therefore, a considerable difference between the two cases of equilibrium represented in figs. 127 and 129. In the former case, if the body be disturbed a little, it falls back again towards its position of rest; it has, in fact, a tendency to steadiness or stability. Equilibrium of this kind is therefore called *stable equilibrium*. In the latter case, if the body be disturbed ever so little from its position of rest, it falls completely away from it; it has, in fact, a tendency to unsteadiness or instability. Equilibrium of this kind is called *unstable equilibrium*.

Fig. 131.

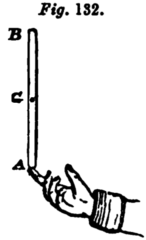
A body, then, is said to be in *stable equilibrium*, if, when it is slightly disturbed from its position of rest, it comes back again to that position: and a body is said to be in unstable equilibrium when, if it is slightly disturbed from its position of rest, it moves away altogether from that position.

A good illustration of stable equilibrium is the common toy represented in fig. 131.

A horse H has a weight W attached to it by means of a bent wire, in the manner shown in the figure, and the hind feet of the horse are placed upon a stand S. The object of the weight W is to make the centre of gravity of the whole body, consisting of the horse, wire, and weight, assume a position vertically below the point of support S, when the horse is in a horizontal position. This is, therefore, a case of stable equilibrium, and if the horse be disturbed a little from the horizontal position, it will always move back towards that position again, and thus imitate the action of galloping by swinging backwards and forwards without falling off the stand.

The difficulty of balancing a long pole, $A B$, on the finger A, fig. 132, is an illustration of unstable equilibrium. The centre of gravity G of the pole, is above the point of support A, and therefore, though the pole will rest where the line $G A$ is truly vertical, it will, on any inclination from the vertical position, fall quite away from that position, unless, by a quick motion of the finger, the point of suppport A is moved about so as to keep it exactly under G. The difficulty of thus balancing the pole on the finger arises from the fact, that the very least disturbance of the pole out of the exact vertical position of equilibrium, causes it to fall quite away from that position, unless the point of support be moved in such a way as to arrest the fall.

Fig. 132.



A child learning to walk is an illustration of this *attempt to balance* a body, supported on such a

small base as the feet are. The child has to learn by experience, though it be not aware of the fact, that, when it stands, the vertical through its centre of gravity must fall within the limits of its feet; and it takes much time and practice to habituate the muscles to keep the body, by an involuntary effort, always in such a position that the vertical through the centre of gravity may not fall without the limits of the feet. Considering the variety of attitudes into which the body is thrown in actions of standing, walking, running, &c., it is not surprising that some practice is necessary to enable a child to fulfil, in an involuntary manner, the mechanical law we are here speaking of, and so avoid falling.

Walking on stilts is much more difficult than on the feet, because the base is much more circumscribed, especially on each side, so that a very little inclination to the right or left is enough to cause a fall, unless the body be thrown quickly in the opposite direction.

GENERAL PRINCIPLE.

We may lay down the following as a *general principle*, namely, *That the equilibrium of a body is stable or unstable, according as its centre of gravity is in its lowest or highest position.*

For if the centre of gravity be in its lowest position, and if the body be slightly disturbed from that position of rest, the centre of gravity has risen in consequence of that disturbance. Therefore, since the centre of gravity always falls if it can, the body, if left to itself, will go back to the position of rest.

But if the centre of gravity be in its highest position, and the body be slightly disturbed from that position of rest, the centre of gravity has fallen in consequence of that disturbance. Therefore, since the centre of gravity never rises of itself, but falls if it can, the body will not return to its position of rest, but will move away from it.

When, after a slight disturbance from a position of rest, a body has no tendency to move, either back to that position or away from it, the equilibrium is neither stable nor unstable, and is therefore called *neutral equilibrium*. An instance of this kind of equilibrium is a suspended body, in which the centre of gravity itself is the point of suspension.

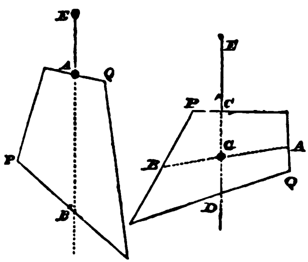
We shall presently, and on future occasions, give other instances of stable and unstable equilibrium.

We may here observe, that the various instances we have been considering, may be explained without any reference to the principle of the descending tendency of the centre of gravity. Thus, in the case represented in fig. 127, the weight acts along the line $C\,G$, because that line is vertical, and therefore, since C is fixed, no motion can ensue. Again, in fig. 128, a vertical through G falls on the left of C, and therefore the weight will have a certain *moment* (see Art. 63.) about C, tending to turn the body towards the right hand. And so in the other cases.

EXPERIMENTAL METHOD OF FINDING THE CENTRE OF GRAVITY.

The fact, that when a body is suspended by any point, the centre of gravity rests vertically below the point of suspension, enables us to determine by actual trial the position of the centre of gravity in the body. Thus, suppose that we wish to find whereabouts the centre of gravity is in a piece of thin board of any shape, as, for instance, *P Q*, fig. 133. Suspend the board by a string *A B*, fastened to it at any point *A*, and mark on the board with a piece of chalk the direction *A B* of the string *E A* produced. *A B* is then a vertical line through the point of suspension, and therefore the centre of gravity must lie somewhere in *A B*.

Fig. 133. *Fig.* 134.

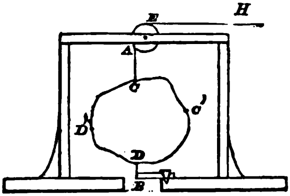


Again, suspend the body by some other point *C*, fig. 134, and mark with chalk on the board the direction *C D* of the string *E C* produced. *C D* is then another line in which the centre of gravity lies, and therefore the point where the two lines *A B* and *C D*, thus drawn, intersect, must be the centre of gravity of the board.

To determine in this manner the centre of gravity of a body, not flat, like the board, but of some solid irregular shape, is, of course, not quite so easy, but it may be done in the following manner:—

Get an upright rame, fig. 135, with a mall fixed pulley E at he top of it; fasten a tring to any point C of he body whose centre f gravity we wish to nd, and suspend the ody by passing the

Fig. 135.



tring over the pulley E, as shown by the line $'AEH$ in the figure. Let B be an upright oint, fixed in the base of the frame exactly nder A, in a vertical line, so that the line AC roduced may pass through B. The proper posi-ion of B may be easily determined, by fastening piece of lead to a string, and allowing it to hang rom A below the base of the frame, making a ole in the base, if necessary, for the string to ass through. It will then be easy to fix a piece f pointed wire, BK, in the base, so that the point B may be in contact with the string; in which case he two points B and A will be in the same ver-ical line.

This being done, and the body being suspended y the string as shown in the figure, we have only o lower the body gradually till it touches and is iarked by the point B; then it is clear that the ne joining C and B is a vertical line drawn hrough the point of suspension, and therefore the entre of gravity must lie somewhere in this line.

If then D be the place on the surface of the ody where the mark was made by the point B, nd if D' be the mark made when the body is sus-ended by any other point C', it is clear that the entre of gravity is somewhere in the line joining

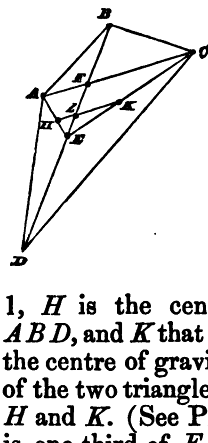
C and *D*, and also somewhere in the line joining *C′* and *D′*; that is, the centre of gravity is at the point where these two lines intersect. In this manner the position of the centre of gravity is completely defined and determined.

PROBLEMS RELATING TO THE FINDING OF THE CENTRE OF GRAVITY.

Problem XIX.

To find the centre of gravity of a quadrilateral figure.

Fig. 136.



Let *ABCD*, fig. 136, be the quadrilateral figure; draw its two diagonals *AC* and *BD*, intersecting at *F*; find *E*, the middle point of *BD*, and draw *AE* and *CE*; take on these lines the points *H* and *K*, so that *EH* shall be one-third of *AH*, and *EK* one-third of *CK*, and draw *HK* cutting *BD* at *L*.

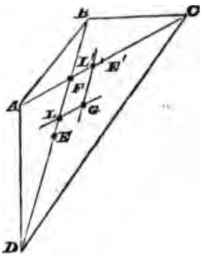
Then, by Prop. XXI. Cor. 1, *H* is the centre of gravity of the triangle *ABD*, and *K* that of the triangle *BCD*: wherefore the centre of gravity of the whole figure composed of the two triangles is somewhere in the line joining *H* and *K*. (See Prop. XIX.) Now, because *EH* is one-third of *EA*, and *EK* one-third of *EC*, it follows that *HK* is a line parallel to *AC*, and that *EL* is one-third of *EF*.* The line *HK* may

* Euclid, Book VI. Prop. 2.

'e be easily drawn as follows:—Find E dle point of BD, and then L, so that EL one-third of EF; then draw the line HK L *parallel* to AC, and the centre of gravity quadrilateral figure is somewhere in this

e we may determine the centre of gravity following simplified construction, fig.

Fig. 137.



ng drawn the diagonals l BD meeting in F, find dle points, E and E', of id determine the points L' by making EL one- f EF, and $E'L'$ one- f $E'F$; then through a line parallel to AC, 'ough L' a line parallel and the point G where ies meet is the centre of required.

is obvious, since, as we have shown, the f gravity must be somewhere in the line L parallel to AC, and, by a parity of ig, it must also be somewhere in the line L', parallel to BD; wherefore it is at it of intersection of the two lines.

l.—If $AB=BC$, and $AD=DC$, where is tre of gravity of the quadrilateral figure), which in this case is called *a lozenge?*

2.—If the diagonals be at right angles to ther, and if $AF=4$, $FC=10$, $FB=9$, 1; find how far the centre of gravity is

3.—On the same supposition, except that

$AF=1$, $FC=2$, $FB=3$, and $FD=4$; find how far the centre of gravity is from F.

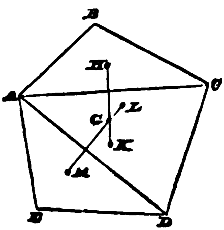
Ex. 4.—On the same supposition as in Ex. 2, except that the diagonals are inclined to each other at an angle (BFC) of 60°; find how far G is from F.

Ex. 5.—If the diagonals be at right angles to each other, and the perpendicular distance of the centre of gravity from one diagonal is 4, and from the other 7; draw the quadrilateral figure.

Problem XX.

To find the centre of gravity of a pentagonal figure.

Fig. 138.



Let $ABCDE$, fig. 138, be the pentagonal figure; find the centre of gravity, K, of the quadrilateral figure $ACDE$; find also the centre of gravity, H, of the triangle ABC, and join H and K. Then the centre of gravity of the whole figure must lie somewhere in the line HK.

Again, find L and M, the respective centres of gravity of the quadrilateral figure $ABCD$, and the triangle ADE, and join L and M. Then the centre of gravity of the whole figure must lie somewhere in the line LM. Wherefore the point G, where LM and HK intersect, is the centre of gravity required.

Ex. 1.—Find the centre of gravity of a pen-

tagon whose sides and angles are all equal to each other.

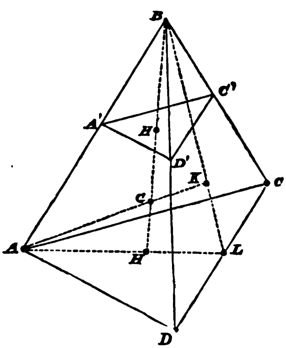
Ex. 2.—AE, ED, DC, and AC are all equal to each other, AB is equal to BC, and the angle AED is 60°; find the centre of gravity of the figure.

Problem XXI.

To find the centre of gravity of a triangular pyramid.

A triangular pyramid is a solid figure, $ABCD$, fig. 139, bounded by four triangular surfaces or faces, namely, ABC, BCD, BDA, and ACD. Observe, in the figure the points A, C, and D are not supposed to lie in the same plane with B. The conception of figures of this kind being rather difficult to those who have never studied the Eleventh Book of Euclid, or anything relating to *solid geometry*, it will be well for a beginner to have a model to show this figure in its natural solid dimensions. Such a model can be easily made by drawing a triangle, ACD, upon a piece of board, making holes at A, C, and D, into which three stout wires, BA, BC, and BD, may be inserted; and these wires are to be united at B. The other lines of the figure, such as BL, BH, $A'B'$, &c., may be formed of thin wire, or of threads.

Fig. 139.



To find the centre of gravity of this solid figure, let H, the centre of gravity of the triangle ACD, be found, and join BH; then we may show that the centre of gravity of the solid figure is somewhere in the line AH. For, if $A' C' D'$ be a triangle formed by lines, $A'C'$, $C'D'$, $D'A'$, parallel to AC, CD, DA, respectively; in other words, if $A' C' D'$ be a triangular *section* * of the solid parallel to ACD, it is clear that the triangles $A'C'D'$ and ACD will be perfectly similar to each other, and similarly situated with respect to the line BH, which is supposed to meet the triangle $A'C'D'$ at H'; wherefore, since H is the centre of gravity of ACD, H' will be the centre of gravity of $A'C'D'$. Now we may suppose the whole solid to be made up of triangular *slices*, such as $A' C' D'$, all parallel to ACD, and the centre of gravity of each of these slices, like that of $A'C'D'$, will lie in the line BH. Consequently, the centre of gravity of the whole solid must lie somewhere in the line BH.

Again, if we find K the centre of gravity of the triangle BCD, and draw the line AK, we may show, in the same manner, that the centre of gravity of the solid must lie somewhere in the line AK.

Wherefore, if G be the point of intersection of these two lines, BH and AK, the centre of gravity of the solid is at G.

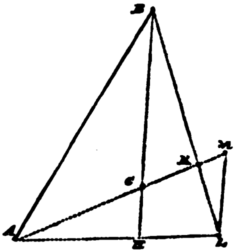
We may show by measurement that HG is always *one-fourth* of HB, by constructing the figure shown in fig. 140, where the letters signify the same points as in fig. 139. L is the middle point of DC; HL is one-third of AL, which

* That is, the flat surface produced by *cutting* the solid figure *across* in one plane, supposed to be parallel to the plane ACD.

:es H the centre of gravity of ACD; and is one-third of BL, :h makes K the centre :avity of BCD. Then, : measure, we shall find HG is always one-:h of HB.

his may be shown by id, Book VI., as follows: raw LM parallel to to meet AK produced f: then, because ML rallel to HG, we have,

Fig. 140.



$$ML : GH :: AL : AH :: 3 : 2.$$

, because ML is parallel to BG, we have,

$$BG : ML :: BK : KL :: 2 : 1.$$

:refore, by composition of ratios,

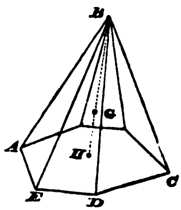
$$BG : GH :: 3 : 1.$$

nce it follows that GH is one-fourth of BH.

rollary 1.—*To find the centre of gravity of a nid on a polygonal base.*

:t $ACDEF$, fig. 141, be the polygonal base, sponding to the triangular in ACD in fig. 139; then nay show that if we draw a BH, from B to the centre avity H of the base $ACDEF$, nake HG equal to one-fourth IB, G is the centre of ty of the pyramid. In fact, nay suppose the pyramid to

Fig. 141.



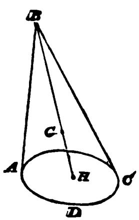
ade up of a set of triangular pyramids, namely,

ABCD, *ABDE*, and *ABEF*; and the centre of gravity of each of these pyramids will be at a distance above the base equal to one-fourth of the distance of the vertex *B* above the base. Wherefore the centre of gravity of the whole solid must be also one-fourth of the same distance above the base. Also we may show, as in the preceding case, that the centre of gravity is somewhere in the line *BH*. It follows, therefore, that if we measure on *HB* a distance *HG* equal to one-fourth of *HB*, *G* must be the centre of gravity required.

Corollary 2.—*To find the centre of gravity of a cone.*

When the polygon *ACDEF*, fig. 141, has a very great number of very small sides, it may be regarded as a curve; in such a case the pyramid becomes what is called a *cone*, fig. 142. We have, therefore, the same rule as before for finding *G*, the centre of gravity of a cone; namely, find *H*, the centre of gravity of the base *ACDE*, draw *BH*, and measure *HG* equal to one-fourth of *BH*.

Fig. 142.



If *ACDE* be a circle, *H* is, of course, its centre.

CENTRES OF GRAVITY OF CURVED FIGURES.

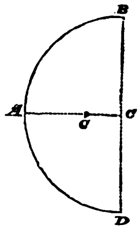
It is often important to find the centres of gravity of certain figures bounded by curved surfaces, but the investigations and rules for this purpose are too difficult to be introduced here;*

* These rules are proved geometrically at the end of the *chapter.*

ll therefore simply state a few results which ; useful.

?*AD*, fig. 143, be a rod bent 1e shape of a semicircle, its of gravity is thus found. Let he centre of the semicircle, and middle point of the semicir- ircumference *BAD*; or, in other let *CA* be perpendicular to then the centre of gravity is line *AC*, and its distance *GC* ' is got by dividing $2AC$ by umber 3.14159; or, what is the same thing, *GC* is got by dividing y 11.

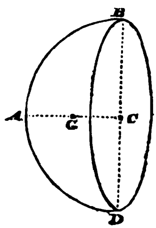
Fig. 143.



number 3.14159 is that decimal* by which lius of a semicircle must be multiplied in to get the length of the semicircular cir- ence; or, we may say, 3.14159 is the number .ch the diameter of a circle must be multi- ɔ get the length of the circumference. The n $\frac{22}{7}$ is nearly the same thing as this number.

'*AD* represent a board in the shape of a :cle, the centre of gravity, found by measuring *CG*, to two-thirds of the value preceding case; that is to '*G* is got by dividing $7AC$

Fig. 144.



'*AD* represent half a sphere, 4, *C* the centre, *A* the middle of the hemispherical surface, eing perpendicular to *BD*,

* To five places of decimals.

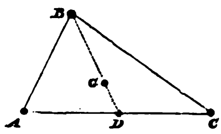
and if G be the centre of gravity; then CG is exactly $\frac{3}{8}$ths of AC.

If G be the centre of gravity of the surface of this half sphere, CG is exactly equal to $\frac{1}{2}AC$; that is, G is the point of bisection of the radius AC.

Problem XXII.

To find the centre of gravity of three equal weights placed at the angular points of a triangle; also, of three rods of equal weight forming a triangle.

Fig. 145.



Let A, B, and C, fig. 145, be three weights, each equal to W, at the angular points of the triangle ABC; bisect AC in D, draw BD, and on it take G, so that DG shall be *one-third* of DB, and therefore *one-half* of BG. The weights A and C may be supposed to be collected into the point D, which is their centre of gravity; we shall then have a weight $2W$ at D, and W at B. But, since $BG = 2DG$, G is the centre of gravity of these weights; and therefore G is the centre of gravity of the three equal weights at A, B, and C.

It appears from this that the centre of gravity of the three weights is coincident with that of the triangle ABC.

If AB, BC, and AC, be three rods of equal weight, their centre of gravity is at the same point G, for the weight of AC, say W, acting at D the middle point of AC, is equivalent to $\frac{1}{2}W$ at A and $\frac{1}{2}W$ at C; in like manner, the weight of AB is equivalent to $\frac{1}{2}W$ at A, and $\frac{1}{2}W$ at B, and the weight of B C is equivalent to $\frac{1}{2}W$ at B

and $\frac{1}{2}$ *W* at *C*; wherefore, the weights of the three rods are equivalent to *W* at *A*, *W* at *B*, and *W* at *C*; and therefore the centre of gravity is at the same point as before.

Ex. 1.—Find the centre of gravity of the weights *W* at *A*, 2 *W* at *B*, and 3 *W* at *C'*.

Ex. 2.—Find the centre of gravity of the rods *A B*, *B C*, and *A C*, supposing their weights to be *W*, 2 *W*, and 3 *W* respectively.

Ex. 3.—Find the centre of gravity of the rods *A B*, *B C*, and *A C*, supposing them to be the same substance and thickness, and that their lengths are $A B = 4$, $B C = 5$, $A C = 6$.

Ex. 4.—Find the centre of gravity of the weights *W*, 2 *W*, 3 *W*, and 4 *W*, placed at the four corners of a square.

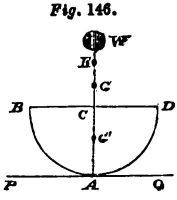
Ex. 5.—Find the centre of gravity of four equal weights, placed at the corners of a quadrilateral figure.

Ex. 6.—Find the centre of gravity of four equal weights, placed at the corners of a triangular pyramid.

Problem XXIII.

If B A D, fig. 146, be a half sphere, whose weight is W, C its centre, and C E a rigid line, by which a weight equal to W is fixed to the half sphere, C E being at right angles to B D; and if this compound body be placed on a horizontal plane P Q; to determine whether the equilibrium is stable or unstable.

Fig. 146.



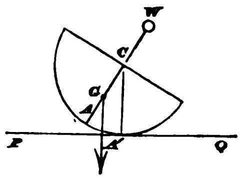
Produce the line *E C* to *A*, and let *G'* be the centre of gravity of the half sphere, which will

be on the line CA, at a distance from C equal to $\frac{3}{8} AC$. We then have a weight W at G', and W at E, and the centre of gravity of these two weights will be at G, which is half-way between G' and E. G may, according to the length of EC, fall either above or below C.

This compound body will rest when the point A is placed in contact with the horizontal plane PQ; for there the point G, when the weight of the whole acts, will be vertically over the point of support.

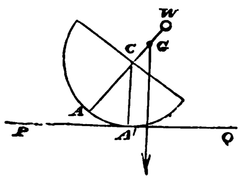
Now, let us suppose that the body is disturbed from its position of rest, in the manner represented in fig. 147, and that G is below C. The force which acts on the body is $2W$, vertically downwards at G, and this force acts on the left of the point A', where the surface BAD touches the horizontal plane. Wherefore, it is evident, that the effect of $2W$ thus acting will be to make the body turn, or rather roll back to its original position; inasmuch as the tendency of W is to bring A down to the horizontal plane again.

Fig. 147.



But if G be above C, as is represented in fig. 148, the reverse will be the case, for then the force $2W$ will act on the right of A', and therefore tend to make the body turn or roll away from its position of rest.

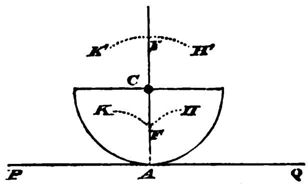
Fig. 148.



ıerefore, the equilibrium is stable or unstable ling as G is below or above C.

we place the body with A touching the ıntal plane, fig. ınd make it roll e right and left, l be found that nt F below C bes a curve like , and that a point ove C describes re like $K' F' H'$.

Fig. 149.



ıerefore, if the centre of gravity be *below* C, F, it will *rise* when the body is disturbed, ' it be above C, as at F', it will *fall;* in the r case the centre of gravity is in its *lowest* on; in the latter, in its *highest*. This agrees the principle, that the equilibrium is stable stable according as the centre of gravity is lowest or highest position.

the centre of gravity be at C, it will de- a straight line horizontally, when the body led right or left. In this case the equi- m is neutral; in fact, A' will always be ally under the centre of gravity, and there- in whatever position the body is placed, it est.

. 1.—If $E\,C = A\,C$, determine whether the brium is stable or unstable.

. 2.—If $EC = \frac{1}{4} A\,C$, determine whether the brium is stable or unstable.

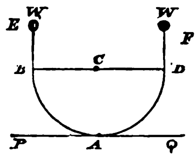
. 3.—Find what must be the length of $E\,C$ ler that the equilibrium may be neutral.

. 4.—If the weight W, which is at E, be ourth *that of the* half sphere, how long may

EC be made without the equilibrium becoming unstable?

Ex. 5.—If two equal weights, W and W, fig. 150, be fixed to the half sphere by equal rigid lines, EB and FD, both perpendicular to BD; and if each weight be equal to the weight of the half sphere, find how long EB and FD may be made without the equilibrium becoming unstable.

Fig. 150.



Ex. 6.—On the same supposition, only that EB is double FD, find how long EB may be made without the equilibrium becoming unstable.

Ex. 7.—If $EB = FD$, but the weight at E is double that at F, the latter being equal to the weight of the half sphere; find in what position the half sphere will rest, and how long EB and FD may be made without the equilibrium becoming unstable.

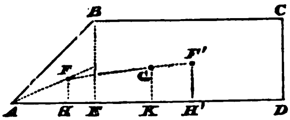
Ex. 8.—On the same supposition, except that the weight at E is four times that at F; find the same.

Ex. 9.—On the same supposition, find how much the weight at E must exceed that at F, and how long EB and FD must be, in order that the half sphere may touch the horizontal plane at a point half-way between A and B, and rest in that position in neutral equilibrium.

Problem XXIV.

If ABCD be a piece of board, fig. 151, *in the shape of a quadrilateral figure, of which two*

Fig. 151.



sides, $A\,D$ and $B\,C$, are parallel, and $C\,D$ at right angles to them; to find the centre of gravity G by means of Prop. XXVII.

Draw $B\,E$ at right angles to $A\,D$; let F be the centre of gravity of the triangle $A\,B\,E$, F' that of the rectangle $B\,C\,D\,E$, and draw $F\,H$, $F'\,H'$, $G\,K$, at right angles to $A\,D$. Then, because F is the centre of gravity of the triangle $A\,B\,E$, and $F\,H$ parallel to $B\,E$, $A\,H$ must be *two-thirds* of $A\,E$; also, H' is the middle point of $E\,D$, for a similar reason.

Let us now assume, for brevity, a to represent $A\,E$, b to represent $E\,D$, and x to represent $A\,K$; then we have,

$$A\,H = \tfrac{2}{3}\,a,\ A\,H' = a + \tfrac{1}{2}\,b.$$

Also, since the triangle $A\,B\,E$, and the rectangle $B\,C\,D\,E$, have the same altitude $B\,E$, if W be the weight of the former, and W' that of the latter, we have,*

$$W : W' :: \tfrac{1}{2}\,a : b.$$

Now, by Prop. XXVII. we have,

$$x = \frac{W \times \tfrac{2}{3}a + W' \times (a + \tfrac{1}{2}b)}{W + W'};$$

or, putting for W and W' the proportional quantities, $\tfrac{1}{2}\,a$, and b, we find,

* The area of (or quantity of surface in) $A\,B\,E$ is $\tfrac{1}{2}A\,E \times B\,E$, and the area of $B\,C\,D\,E$ is $E\,D \times B\,E$, and W and W' are proportional to these respectively; wherefore,

$$W : W' :: \tfrac{1}{2}A\,E \times B\,E : E\,D \times B\,E :: \tfrac{1}{2}A\,E : E\,D.$$

$$x = \frac{\frac{1}{3}a^2 + b(a + \frac{1}{2}b)}{\frac{1}{2}a + b}.$$

This determines the value of x, or AH', in terms of a and b, and thus the position of the line KG becomes known.

If we bisect BC and AD, and join the points of bisection, the joining line will bisect every line drawn in the quadrilateral figure, parallel to AD or BC; wherefore, the centre of gravity G must lie somewhere in the joining line; therefore G is at the intersection of that line and $H'G$. Thus G is determined.

Ex. 1.—If $AE = ED = DC = 10$, find the position of H', and determine G by construction.

Ex. 2.—If $AE = 2ED = 10$, determine H' and G.

Problem XXV.

To find the proportionate dimensions of the board in the last problem, so that, when suspended from the point B, it may hang with AD in a horizontal position.

In this case G must be vertically below B, and therefore must lie in the line BE; consequently, H' must coincide with E, and therefore $x = a$. We have then, by the preceding problem,

$$a = \frac{\frac{1}{3}a^2 + b(a + \frac{1}{2}b)}{\frac{1}{2}a + b};$$

$$\therefore (\tfrac{1}{2}a + b)a = \tfrac{1}{3}a^2 + b(a + \tfrac{1}{2}b);$$

$$\therefore 3a^2 + 6ab = 2a^2 + 6ab + 3b^2;$$

$$\therefore a^2 = 3b^2, \text{ or } a = \sqrt{3}\,b = 1.732\,b.$$

Hence, $A E$ must be equal to $E D$ multiplied by 1.732, and this determines the necessary proportional dimensions of the board.

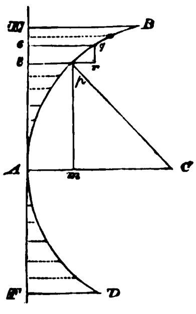
GEOMETRICAL INVESTIGATION OF RULES FOR FINDING THE CENTRE OF GRAVITY, IN THE CASE OF BODIES BOUNDED BY CURVED SURFACES.

PROPOSITION XXVIII.

To find the moment of the weight of a rod, in the form of a circular arc, about the centre of the circle as fulcrum.

Let $B A D$, fig. 152, be the rod bent into the form of a circular arc, whose centre is C; take A any point of the rod, and draw $A C$, which we shall suppose to be horizontal; draw $E F$ through A at right angles to $A C$; also, $E B$ and $F D$ parallel to $A C$. Let $p q$ be any small portion of the rod, so small that we may regard it as a straight line without sensible error; draw $p C$; $p m$ at right angles to $A C$; $t r$ through p, and $s q$, both parallel to $A C$; and $q r$ at right angles to $p r$.

Fig. 152.



Then $p q$ is at right angles to $p C$, because the circumference of a circle always runs at right angles to the radius; also, $p r$ and $q r$ are respectively at right angles to $p m$ and $m C$. Wherefore, the triangles $q p r$ and $p m C$ are similar, and we have, by Euclid, Book VI., observing that $t s = q r$, and $p C = A C$,

$$pq : qr :: pC : mC;$$

and therefore $pq \times mC = ts \times AC$.

Now, suppose that EF is a rod of the substance and thickness of the bent rod BAD, in which case we may take the *lines* pq and ts to represent the *weights* of these two small portions of the rods respectively. Then, $pq \times mC$ is the moment of the weight of the portion pq, with reference to C as fulcrum; because this weight is a force acting along the vertical direction pm,* and mC is the perpendicular from C on pm. Also, for similar reasons, $ts \times AC$ is the moment of the weight of the portion ts, but $pq \times mC = ts \times AC$; wherefore, it appears, that the moment of pq about C is equal to that of ts about C.

Now, if we divide the whole of the rod BAD into small portions, such as pq, and if we also divide the whole of the rod EF into corresponding portions, such as ts, by drawing horizontal lines, as shown in the figure; we may prove, as in the case of pq and ts, that the moment of each portion of BAC about C, is equal to the moment of the corresponding portion of EF about C. Wherefore it follows, that the moment of the whole weight of the curved rod BAD about C as fulcrum, is equal to the moment of the weight of the straight rod EF about C.

Since the moment of EF is the weight of EF multiplied by AC, this is also the moment BAD. *Which was to be found.*

* Strictly speaking, this vertical ought to be drawn through the middle point of pq; but pq is so small, that the error arising from supposing this vertical to coincide with pm is quite insensible.

Corollary.—If W be the weight of $B\,A\,D$, then, since the rods $B\,A\,D$ and $E\,F$ are supposed to be of the same substance and thickness, we have,

$$\text{weight of } E\,F : W :: E\,F : \text{arc}\, B\,A\,D,$$

$$\text{and therefore weight of } E\,F = \frac{W \times E\,F}{\text{arc}\, B\,A\,D};$$

whence the moment of the rod $B\,A\,D$ about C is

$$\frac{W \times E\,F \times A\,C}{\text{arc}\, B\,A\,D}.$$

Proposition XXIX.

If the rod $A\,B$ be supposed to revolve about $A\,C$ as axis, and so generate or describe a portion of a spherical surface; * *it is required to find the moment of that surface about C as fulcrum.*

If we suppose the whole figure $B\,E\,F\,D$ to revolve about $A\,C$ as axis, it is clear that the portions $p\,q$ and $t\,s$ will each describe circular rings of equal radius, one ring being described with $p\,m$ as radius and m as centre, and the other with $A\,t$ as radius and A as centre. Wherefore, the quantity of matter in the ring described by $p\,q$, will be to the quantity of matter in the ring described by $t\,s$, as $p\,q$ is to $t\,s$. The weights of the two rings will therefore be proportional to $p\,q$ and $t\,s$ respectively; and therefore, since $p\,q \,.\, m\,C = t\,s \,.\, A\,C$, it follows that,

$$\begin{aligned}&(\text{weight of ring described by } p\,q) \times m\,C = \\ &(\text{weight of ring described by } t\,s) \times A\,C.\end{aligned}$$

* By the word *surface* here we mean a *material surface*, that is, a body of extremely small and uniform thinness, like paper, for instance.

But these weights act at the centres of gravity of the rings, namely, at A and m respectively, in the vertical directions $t\,A$ and $p\,m$; wherefore $m\,C$ and $A\,C$ are the perpendiculars from C on the directions of these weights. We have, therefore,

moment about C of ring described by $p\,q$ =
moment about C of ring described by $t\,s$.

Wherefore, reasoning as in the former proposition, it appears, that the moment about C of the portion of spherical surface, generated by the revolution of the rod $B\,A$, is equal to the moment about C of the circular surface, generated by the revolution of the rod $A\,E$. *Which was to be found.*

The latter moment is the weight of the circle, whose radius is $A\,E$ multiplied by $A\,C$; therefore, this is also the moment of the portion of a spherical surface generated by $B\,A$. Of course, both surfaces are supposed to be equally thin, and to be composed of the same substance.

Corollary.—If W be the weight of the surface generated by the revolution of the rod $B\,A$, we have, as in the former proposition,

weight of circle generated by rod $E\,A$: W :: surface of circle whose radius is $E\,A$: surface generated by $A\,B$.

Wherefore, the moment about C of the weight of the surface generated by the revolution of the arc $B\,A$ is,

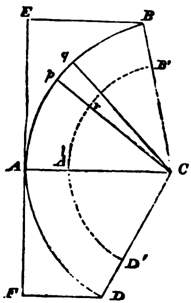
$$\frac{W \times \text{surface of circle whose radius is } E\,A \times A\,C}{\text{surface generated by arc } A\,B.}$$

Proposition XXX.

d the moment of the weight of a flat surface, ape of a sector of a circle, about the centre cle as fulcrum.

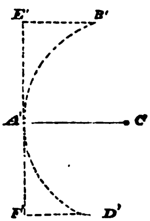
B A D, fig. 153, be the sector, *C* the centre, the horizontal radius; may divide the whole to triangles, by drawing number of radii from *C*, stance, *C p* and *C q*, *p q* nall enough to be re- as a straight line, as

Fig. 153.



if *r* be the centre of of the triangle *Cpq*, *Cr* o-*thirds* of *Cp*, we may the whole weight of be collected into the and the same may be all the other triangles ich the sector may be Now, the centres of of all the triangles form r arc *B' A' D'*, described as centre, with a radius o *two-thirds* of *A C*. ore, we may suppose le weight of the sector

Fig. 154.



llected into the series of points forming *B' A' D'*; in other words, we may suppose tead of the sector, we have a rod *B' A' D'*. f we draw *E' B'* and *F' D'* horizontally, as 54, and *E' F'* through *A'* vertically, the

moment of this rod is the weight of $E'F'$ multiplied by $A'C$, $E'F'$ being supposed to be a rod of the same substance and thickness as $B'A'D'$. Wherefore, this is also the moment of the sector. *Which was to be found.*

Corollary.—If W be the weight of the sector, W is also the weight of $B'A'D'$, and therefore, since $B'A'D'$ and $E'A'D'$ are of the same substance and thickness, we have,

$$\text{weight of } E'F' : W :: E'F' : \text{arc } B'A'D'$$
$$:: EF : \text{arc } BAD.$$

EF being vertical, and EB and FD horizontal;

$$\text{therefore weight of } E'F' = \frac{W \times EF}{\text{arc } BAD}.$$

Whence, since $A'C' = \frac{2}{3} AC$, we find that the moment of the sector about C is

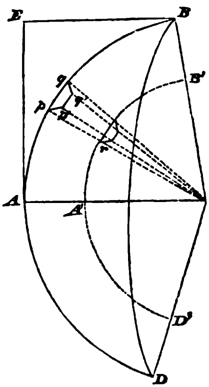
$$\frac{2}{3} \frac{W \times EF \times AC}{\text{arc } BAD}.$$

Proposition XXXI.

To find the moment of a body, in the shape of a solid sector of a sphere, about the centre of the sphere as fulcrum.

Let $ABCD$, fig. 155, represent the solid sector, and C its centre; let $pqq'p'$ be a small portion of the surface of the sphere, so small that we may regard it as a little plane, and draw the lines pC, $p'C$, qC, $q'C$. Then, if the centre of gravity of the prism $pp'qq'C$ be r, Cr will be *three-fourths* of Cp, (Prob. XXI.) and we may, as in the former

Fig. 155.



sition, suppose the it of this prism to be ted into the point r. we may suppose the sector to be divided risms of this kind, and eight of each prism to llected into its centre vity, at a distance from ıal to *three-fourths* of dius of the sphere.

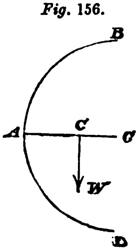
may therefore suppose he whole weight of the sector is collected into es of points forming a cal surface, whose radius *ee-fourths* of that of the sector. Hence, ing as in the former proposition, we obtain llowing expression for the moment of the ector, viz.—

$$\frac{(\text{surface of circle whose radius is } EA) \times AC}{\text{spherical surface of the sector}};$$

ing the whole weight of the solid sector.

Proposition XXXII.

Fig. 156.



find the centre of gravity of cumference of a semicircle.

Prop. XXVIII. the moment semicircle BAD, fig. 156, C, supposing that AB and re each quadrants, in which $AF = AE = AC$, (see fig. vill be,

$$\frac{W \times EF \times AC}{\text{arc } BAD}, \text{ or } \frac{W \times 2AC \times AC}{\text{circumference of semicircle}}.$$

Now, the circumference of the semicircle is got by multiplying its radius AC by 3.14159, or $\frac{22}{7}$ nearly; wherefore, the expression for the moment becomes

$$W \times \frac{14}{22} AC, \text{ or } W \frac{7}{11} AC.$$

Let G be the centre of gravity, which is somewhere on AC; then we may suppose that W acts at G, and therefore the moment of W about C is $W \times CG$. Wherefore,

$$W \times CG = W \times \frac{7}{11} AC;$$

$$\text{and therefore } CG = \frac{7}{11} AC;$$

which determines the position of G.

Proposition XXXIII.

To find the centre of gravity of the surface of a semicircle.

Reasoning as in the preceding case, we find, by Prop. XXX. that the moment of W is

$$\frac{2}{3} \frac{W \times 2AC \times AC}{\text{circumference of semicircle}}, \text{ or } W \frac{14}{33} AC;$$

but the moment of W is also $W \times CG$; wherefore,

$$CG = \frac{14}{33} AC;$$

which determines G.

Proposition XXXIV.

To find the centre of gravity of the surface of a hemisphere, or of the solid hemisphere itself.

In the case of the surface, we have by Prop. XXIX.

$$\text{moment of } W =$$

$$\frac{W \times \text{surface of circle whose radius is } AC \times AC}{\text{surface of hemisphere}};$$

but the surface of the hemisphere is double the surface of the circle, by a well-known geometrical theorem;* wherefore,

$$\text{moment of } W = W \times \tfrac{1}{2} AC = W \times CG,$$

$$\text{and consequently, } CG = \tfrac{1}{2} AC.$$

Again, in the case of the solid hemisphere itself, by Prop. XXXI.

$$\text{moment of } W =$$

$$\frac{3}{4} \frac{W \times \text{surface of circle whose radius is } AC \times AC}{\text{surface of hemisphere}}$$

$$= \frac{3}{4} W \frac{AC}{2} = W.CG;$$

Wherefore, $CG = \frac{3}{8} AC$.

To these Propositions we shall add the following, which is usually given in this part of the subject.

* The surface of a sphere is four times that of its generating or *great* circle. This is proved at the end of the chapter, by the *Properties of Guldinus.*

Proposition XXXIV. (*bis.*)

If a body having a spherical base be placed resting on the top of a sphere, or in the inside of a sphere, to determine whether the equilibrium is stable or unstable.

Fig. 157.

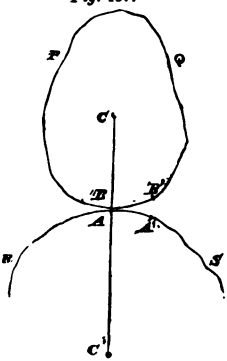
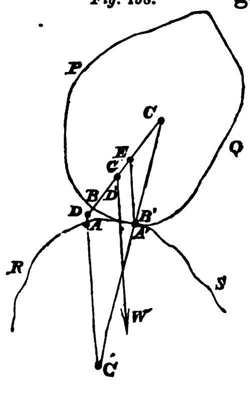


Fig. 158.



Let BPQ be the body with the spherical base, and RAS the upper part of the sphere upon which it rests; let C be the centre of BQP, and C' the centre of RAS, then the line joining C and C' must pass through the two touching points A and B of the two spherical surfaces; also, in order that the body BPQ may rest in this position, the line BC must be vertical, and the centre of gravity of BPQ must be somewhere in BC or BC produced. Our object is to determine how high the centre of gravity may be above B without the equilibrium becoming unstable.

Let the body BPQ be turned or rolled a little on one side, so as to bring, for instance, the point B' upon the point A', as is shown in fig. 158. The

line joining C and C' will now pass through these two touching points, as shown in the figure. Produce CB and $C'A$ to meet at D, and draw $A'E$ vertically to meet CB at E.

Now the centre of gravity of the body BPQ is somewhere in the line BC, or BC produced, as we have stated. If it be below the point E, at G for instance, the weight W of the body will act on the left of the point A', and therefore tend to make the body roll back to its position of rest. But if G fall above E, the reverse will be the case. Wherefore, the *condition of stability is, that BG shall be less than BE.*

Now to find BE, we have, since $A'E$ is parallel to $C'D$,

$$CE : CD :: CB' : CC'.$$

Let us take r to represent BC, and r' to represent AC'; then $CB' = r$, and $CC' = r + r'$; also, because the body is supposed to be but very little disturbed from its position of rest, we may, without sensible error, neglect the little space BD, and assume that $CD = CB = r$. Hence the proportion becomes,

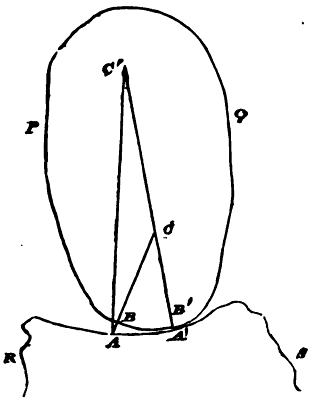
$$CE : r :: r :: r + r'$$

$$\therefore\ CE = \frac{r^2}{r + r'}$$

$$\text{and } \therefore\ BE = r - CE = \frac{rr'}{r + r'}.$$

Hence, if the height of the centre of gravity G above B be less than the product of the two radii, r and r', divided by their sum, the equilibrium

Fig. 159.



will be stable; otherwise not.

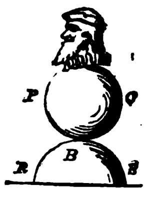
If the body be placed inside a sphere instead of upon it, a figure, apparently different, must be drawn, fig. 159, but the letters denote the same as before, and the above proof applies, word for word, to this case; only we find $CC' = r' - r$, and then, proceeding as above, we obtain,

$$CE = \frac{r^2}{r' - r},$$

$$\therefore BE = r - CE = \frac{rr'}{r' - r}.$$

The only alteration, therefore, is, that we have the *difference* of the radii instead of the *sum*.

Fig. 160.



A good toy is made on the principle proved in this proposition: a half sphere of wood, *RS*, fig. 160, is laid on a table, and a figure, *PQ*, having a heavy spherical base at *B*, is placed upon it. The base of the figure is made so heavy, that the centre of gravity is sufficiently low to render the equilibrium stable. Then, if we push the figure, it will not fall off from *RS*, but roll backwards and forwards on the top of it.

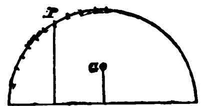
We shall conclude the chapter with the two propositions usually called the Properties of Guldinus.

PROPERTIES OF GULDINUS.

Proposition XXXV.

If AB, fig. 161, be a straight ne, APB any curve, and G e centre of gravity of this rve; then the quantity of surce generated by the revolution the curve APB about the e AB as axis, is found by ultiplying the length of the curve APB by the space hich its centre of gravity G goes over during the volution.

Fig. 161.



Divide the whole of the curve APB into very nall portions, and let P be one of them; draw M and GH at right angles to AB, and let PM represented by y, GH by a, and the length of e small portion P of the curve by s; also let c present the circular arc which G describes when e whole figure is supposed to revolve about AB axis. Then we have,

$$\text{arc described by the point } P : c :: y : a,$$

$$\therefore \text{ arc described by } P = \frac{cy}{a}.$$

Now, the quantity of surface in the narrow ng or belt which the small portion s describes aring the revolution, will evidently be found by ultiplying s by the circular arc which each point

of it describes;* that is, by $\frac{cy}{a}$: wherefore,

$$\text{quantity of surface generated by } s = \frac{cys}{a}.$$

In like manner, if s', s'', s''', &c. denote the other small portions of the curve, and y', y'', y''', &c. their respective perpendicular distances from the axis AB, the quantities of surface generated by these portions will respectively be,

$$\frac{cy's'}{a}, \frac{cy''s''}{a}, \frac{cy'''s'''}{a}, \text{\&c.}$$

Wherefore the whole quantity of surface generated by the revolution of the curve APB will be the sum of all these; that is,

$$\frac{c}{a}(ys + y's' + y''s'' + y'''s''' + \text{\&c.})$$

But by Prop. XXVII. the whole expression in the brackets is equal to

$$(s + s' + s'' + s''' + \text{\&c.})\,a;$$

for we may consider s, s', s'', &c. as representing the weights of the different portions of the curve; wherefore, since $s + s' + s'' + s''' + \text{\&c.}$, is the whole length of the curve, we find that the surface generated by the revolution of the curve is equal to

$$\frac{c}{a}(s + s' + s'' + s''' + \text{\&c.})\,a, \text{or } c \times \text{length of curve.}$$

Which was to be proved.

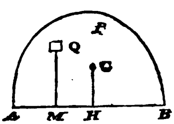
* We consider s to be so small, that every point of it may be regarded as at the same distance y from the axis AB, and therefore all the points of s may be considered as describing circular arcs of the same length.

Proposition XXXVI.

G be the centre of gravity, not of the curve ?, but of the flat surface enclosed by the curve ? and the line AB, the bulk of the solid, gene- by the revolution of this surface about AB as will be found by multiplying this surface by the the centre of gravity G goes over.

: shall use exactly the same notation as :, except that we shall suppose the whole of ırface APB to be divided into very small ns, of which Q is one, fig. 162, 'e shall assume s to represent uantity of surface in Q, and represent the perpendicular ce QM of Q from AB. We also assume $s' s'' s'''$, &c. to

Fig. 162.



e the other portions, corresponding to Q, which the surface APB is divided, and $'''$, &c. to denote their respective perpen- ır distances from AB. Then, reasoning :y as before, we may show that the length of rc described by each point of the small sur- is $\frac{cy}{a}$; but the bulk of the solid ring gene- by s will evidently be found by multiplying the circular arc which each point of it de- s; that is, $\frac{cy}{a}$; wherefore,

$$\text{bulk of solid generated by } s = \frac{cys}{a}.$$

nce, as before, the bulk of the solid gene- by the whole surface APB will be

$$\frac{c}{a}(ys + y's' + y''s'' + y'''s''' + \&c.)$$

or, by Prop. XXVII.,

$$\frac{c}{a}(s + s' + s'' + s''' + \&c.)\, a,$$

or $c \times$ quantity of surface in APB,
which was to be proved.

Examples of the use of these Propositions.

Ex. 1. *To find the surface of a sphere.*—Let the curve APB be a semicircle; then the surface it generates will be that of a sphere whose radius is $\frac{1}{2}AB$, which call r. In this case, $c = \frac{7}{11}r$ by Prop. XXXII. and the circumference of the circle described by G, the radius being c, is $2\,\frac{22}{7}\,c$,* or $4r$. Also the length of APB is $\frac{22}{7}r$. Therefore, by Prop. XXXV.,

surface of sphere $= 4.\,\frac{22}{7}r^2 = 4$, surface of circle whose radius is r.

Ex. 2. *To find the bulk of a sphere.*—By Prop. XXXVI. this bulk is

(quantity of surface in APB) $\times\, c$;

but quantity of surface in $APB = \frac{1}{2}\cdot\frac{22}{7}r^2$.

also, by Prop. XXXIII. in this case,

$$c = 2\cdot\frac{22}{7}\cdot\frac{14}{33}r = \frac{8}{3}r.$$

---

* Using approximate numbers.

Wherefore, bulk of sphere $= \frac{4}{3} \cdot \frac{22}{7} r^3$.

We assume here that the circumference of a circle whose radius is r, is $2 \frac{22}{7} r$, or rather, $2\,(3.14159)\, r$; and that the quantity of surface in the circle is $\frac{22}{7} r^2$, or rather $(3.14159)\, r^2$.

It has been proved in a variety of ways that the ratio of the circumference of a circle to its diameter is about 22 : 7, or, rather, 3.14159 : 1; wherefore, if c be the circumference,

$$c : 2r :: 22 : 7, \text{ and } \therefore c = 2 \frac{22}{7} r.$$

It may also be shown by dividing the circle into a great number of triangles, by drawing radii from the centre, that the quantity of surface enclosed by the circle is found by multiplying half the circumference by the radius; which gives,

$$\text{surface of circle} = \tfrac{1}{2} cr = \frac{22}{7} r^2.$$

CHAPTER VI.

THE MECHANICAL POWERS.

We have now explained what may be regarded as the fundamental part of Mechanical Science, namely, the subject of the composition and resolution of forces under various circumstances; we have dwelt particularly upon the important case of the parallel forces acting on bodies in consequence of the attraction of gravitation, and as connected therewith, we have given a variety of propositions and problems relating to the centre of gravity. We may now proceed to the application of the principles we have established to various mechanical combinations, such as the *simple machines* called the *Mechanical Powers;* the investigation of the *Strains* sustained by different descriptions of *frame-work*, as, for instance, *Roofs* of different kinds; the equilibrium of *rigid structures*, such as *Piers* and *Arches;* the equilibrium of *flexible structures*, such as *Chain Bridges;* the *Strength of Materials;* the effects of *Friction*, &c. Our limits will not allow us to give more than a brief and general view of some of these topics, but even such a view of them will be found useful and instructive.

We shall now commence with the *Mechanical Powers*, as they are called. These are machines,

trivances of the simplest description, which largely into the composition of complex iery, and are of common use and application ordinary concerns and occupations. The nical Powers, as generally enumerated, are —

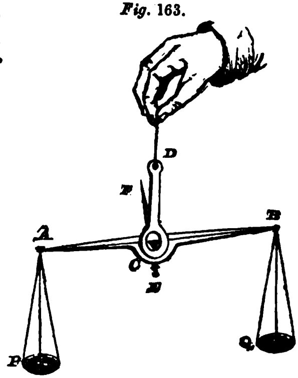
The Lever.
Wheels and Axles.
The Inclined Plane.
Cords and Pullies.
The Screw.
The Wedge.

have already considered the *Lever*, though hat out of its place, for the reasons stated ıpter I. It only remains here to consider *ılance,* which is a description of lever of ərable importance.

THE BALANCE.

ription of the com- ılance.—The com- ıalance consists of n, or lever, *AB*, 3, which is sus- l by its middle *C*, by means of a of metal *CD*. The rms *CA* and *CB* ıal, and from their ıities *A* and *B* are ıded by strings two or scale pans *P* of equal weight.

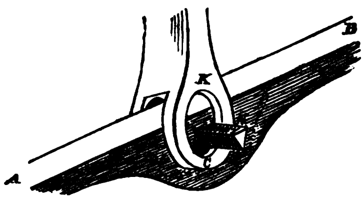
Fig. 163.



In one of these are put the weights, and in the other the substance to be weighed. In many balances there is a projecting piece *E* attached to the arm *A B*, the object of which is to make the centre of gravity of the arm fall below the point of suspension *C*, for reasons we shall presently explain. Of course, this may also be done by giving a proper shape to the arm without the projection *E*. There is also generally a projecting piece *CF*, called a *tongue*, attached to *A B*, and at right angles to *A B*, the object of which is to show whether the beam *A B* is in a horizontal position or not; for, when the tongue *CF* coincides with the vertical suspending piece of metal *CD*, the beam is evidently horizontal.

The Knife Edges.—The beam of the balance has generally what are called *knife edges* projecting from it on each side at *C*, and these knife edges rest on two rings which form the lower part of the suspending piece of metal *C D*. This is shown roughly in fig. 164, where *N* is one knife edge resting in the ring *K*, the other knife edge and ring corresponding being on the other side of the beam. A knife edge is a projecting piece of hard metal in the shape of a small wedge, the lower edge of which is supported on another piece of hard metal, such as the ring *K*. The method of supporting the beam allows it to move about the *point C* with great freedom, much more than

Fig. 164.



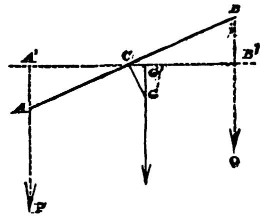
would be if the projecting piece NC were a round pin or axis turning in a circular hole. In fact, there would always be some amount of friction with an axis, which would resist the free motion of the beam; but with a knife edge well made there is practically no friction whatever.

The method of using the common balance is so familiar to every one that we shall not delay to say anything about it.

Proposition XXXVII.

To determine the condition of equilibrium of the common Balance.—Let AB, fig. 165, be the line joining the two points A and B from which the scale pans are suspended; and C the point or fulcrum about which the beam turns, C being in fact the edge of the projecting piece, which, as we have stated, is called the knife edge; then C must be on the line AB, and exactly half way between A and B, for we shall show that this is necessary to constitute a correct balance. Let G be the centre of gravity of the whole beam, including all those appendages, such as the knife edges, the tongue, &c. which are rigidly attached to the beam; then GC is always, and must be for correctness, exactly perpendicular to AB.

Fig. 165.



Let P and Q be the weights in each scale pan, which will be forces acting vertically downwards on the beam, at the points A and B respectively;

also, let W be the weight of the beam and its appendages, which is a force acting vertically downwards at G. These are the forces which balance each other, when the beam assumes its position of rest; for we leave out of account the weights of the scale pans and of the suspending strings, because they are assumed to be exactly equal, and, since they act at points A and B equidistant from the fulcrum, they balance each other.

Let us suppose that AB has assumed its position of rest at a certain unknown inclination to the horizon; and draw through C the horizontal line $A'B'$, meeting the vertical lines PA, WG, QB, produced, if necessary, at the points A', G', and B', respectively.

Now, by the Principle of the Lever, we have,

$$P \times CA' = W \times CG' + Q \times CB',$$

and therefore, $W \times CG' = P \times CA' - Q \times CB'$:

but $CB' = CA'$, evidently, because $CB = CA$; wherefore,

$$W \times CG' = (P - Q)\, C' \quad \ldots\ldots \quad (1)$$

$$\text{or, } CG' = \frac{P - Q}{W}\, CA' \quad \ldots\ldots \quad (2)$$

Either of these equations (1) or (2) is the condition of equilibrium of the balance, *as required*.

Corollary 1.—If $P = Q$, this condition gives $CG' = 0$, and therefore the point G must be vertically under C; for if $CG' = 0$, the points C and G' coincide, and therefore CG is vertical. Hence CG is vertical, and therefore *the beam is horizontal*, when the *weights P and Q in the scale pans are equal*. This is the principle of the com-

mon balance, inasmuch as we conclude that the weights in the scale pans are equal when we see the beam resting in a horizontal position.

Corollary 2.—The condition of equilibrium of the balance is satisfied when $P = Q$, and $W = 0$, no matter what may be the inclination of the beam; for the equation (1) is satisfied when $P = Q$, and $W = 0$, no matter what may be the values of CG' and CA', that is, no matter what may be the inclination of the beam, (for CG' and CA' have different values according to the different inclinations of the beam.) Wherefore, if the weight of the beam and its appendages were nothing, the beam would rest in any position when the weights in the scale pans were equal; and consequently we should not be able to use the balance in the ordinary way, for the beam resting out of the horizontal position would be no indication of the inequality of the weights P and Q.

Corollary 3.—If $CG = 0$, it is clear that CG' also $= 0$; and if at the same time $P = Q$, the equation (1) will be satisfied whatever be the value of CA', that is, whatever be the inclination of the beam. Wherefore, if G coincides with C, the balance cannot be made use of in the ordinary way.

Hence we may see that the weight W of the beam and its appendages is an important part of the balance; but its point of application G must not coincide with the point of suspension C.

Proposition XXXVIII.

To estimate the sensibility of a Balance.—We determine the weight of a body put in one scale pan by putting weights into the other scale pan

until the beam is made horizontal, but it is not generally possible to determine the exact weight with great nicety, in consequence of the small deviation from the horizontal position which the beam undergoes when there is only a trifling difference between the weights in each scale pan. For very nice purposes the balance ought to be so made that an extremely small overplus of weight in either pan should turn the beam out of the horizontal position sufficiently to be obvious to the eye; in other words, the balance should be *sensible* to a very small overplus of weight in either scale pan.

This *sensibility* of the balance consists in the deviation of the beam from the horizontal position, caused by a small given overplus of weight in one of the pans; the greater, therefore, the deviation, the greater the sensibility in proportion. To estimate the sensibility then, let us suppose that W, P, and Q, are expressed in *grains*, and that P exceeds Q by *one grain*, or that $P - Q = 1$; then by the equation (2) in the preceding proposition we find

$$CG' = \frac{P-Q}{W} CA' = \frac{CA'}{W}.$$

But the triangles ACA' and GCG' are evidently similar to each other, inasmuch as CG is perpendicular to AC, CG' to AA', and GG' to CA', wherefore the sides of these triangles are proportional to each other, and we get,

$$AA' : CG' :: CA : CG,$$

$$\text{therefore } AA' = \frac{CG' \times CA}{CG} = \frac{CA' \times CA}{CG \times W}.$$

But because there is but a small overplus, the deviation of the beam from the horizontal will be small, and AC will nearly coincide with $A'C$; we may, therefore, without material error, put CA in place of CA', in the value of AA'. This gives

$$AA' = \frac{(CA)^2}{CG \times W}.$$

Now AA' shows the inclination of the beam; for the greater AA' is, the more apparent will be the inclination of the line CA to the horizontal line CA'; wherefore the *sensibility* of the balance is indicated by the magnitude of AA', that is, by

$$\frac{(CA)^2}{CG \times W}.$$

In fact, this expression gives the depression of the end A of the beam below the horizontal line CA', in consequence of one grain more being in the scale pan suspended from A than in that suspended from B, and by this expression therefore the sensibility of the balance is estimated. *Which was to be done.*

Ex. 1.—Let $CA = 12$ inches, $CG = 1$ inch, and $W = 2880$ grains; then

$$AA' = \frac{(CA)^2}{CG \times W} = \frac{144}{2880} = \frac{1}{20}.$$

Hence the sensibility in this case is measured by the fraction $\frac{1}{20}$, or, to speak more definitely, an overplus of one grain in either scale pan will turn the beam so that one extremity will sink *one-twentieth* of an inch below the horizontal line $A'CB'$, and the other extremity of course rise to the same extent.

Ex. 2.—On the same supposition, except that $CG = \frac{1}{5}$th of an inch, estimate the sensibility.

In this case $AA' = \frac{1}{4}$; that is, an overplus of one grain will make one extremity of the beam sink *a quarter* of an inch. Hence the balance is *five* times more sensible in this case than in the former, inasmuch as $\frac{1}{4} = 5 \times \frac{1}{20}$.

Corollary.—Hence we may see that the shorter CG is, the more sensible the balance is in proportion. It is also manifest from the formula for AA', that the sensibility is increased by diminishing W the weight of the beam and its appendages; it is also increased by lengthening the arms CA and CB.

Proposition XXXIX.

To estimate the Stability of a Balance.

A very sensible balance is by no means convenient for common purposes, because it swings or oscillates so much when disturbed, that it takes too much time in assuming its position of rest. For weighing substances in the ordinary way, it is necessary that the beam should come quickly into the horizontal position when the weights P and Q are made equal, and if this be the case the balance is said to possess the property of *stability;* by which word is meant a tendency to come quickly into a position of rest.

Now this stability will be proportional to the moment of the force which tends to make the beam assume its position of rest. This force is W; for since P and Q are equal they balance each other, and have no tendency to move the beam

one way or the other. Wherefore the stability is proportional to the moment of W, that is, to $W \times CG'$. But we have shown that

$$AA' : CG' :: CA : CG,$$

$$\text{and therefore } CG' = \frac{CG \times AA'}{CA}.$$

Consequently, the stability is proportional to $\frac{W \times CG \times AA'}{CA}$, or $\frac{W \times CG}{CA}$, if we suppose that

$$AA' = 1 \text{ inch.}$$

In other words, this expresses the moment of what we may call *the force of restitution for a depression of one inch;* that is, this expression shows the energy of the force which tends to *restore* the beam to its horizontal position, when one extremity A is depressed one inch below that position. The formula, then, which estimates the stability of a balance is

$$\frac{W \times CG}{CA}.$$

Ex. 1.—If $CA = 12$ inches, $CG = 1$ inch, and $W = 2880$ grains; we have,

$$\frac{W \times CG}{CA} = \frac{2880}{12} = 240.$$

Hence, when the end A of the beam is depressed one inch below its horizontal position, the moment of the force of restitution is 240; that is, the force of restitution is equivalent to 240 *grains* supposed to act at an arm of *one inch.*

Ex. 2.—On the same supposition, except that $CG = \frac{1}{10}$th of an inch, we find that

$$\frac{W \times CG}{CA} = \frac{288}{12} = 24.$$

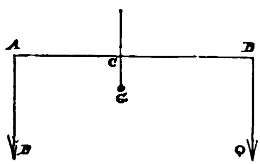
Wherefore the force of restitution is equivalent to only 24 *grains*, supposed to act at an arm of one inch. In this case, therefore, the stability may be said to be $\frac{1}{10}$th of what it was in the former case.

Corollary.—The stability is increased by increasing W and CG, as is evident from the formula. Wherefore, by the Corollary of the previous Proposition, if we thus *increase the stability*, we *diminish the sensibility*, and *vice versâ*.

Proposition XL.

To determine by experiment whether the Arms of a Balance are equal or not.

Fig. 166.



Let AB be in a horizontal position, and therefore G vertically below C, fig. 166; then W will produce no effect, since it acts at G, and we need not take it into account. Wherefore the condition of equilibrium is, by the Principle of the Lever,

$$CA : CB :: Q : P.$$

Hence, if CA be greater than CB, the weight P suspended from A must be less than the weight Q suspended from B, and *vice versâ*.

Suppose now that CA is greater than CB, then

the weight P suspended from A must be less than the weight Q suspended from B. Let us reverse these weights, suspending P from B, and Q from A; then the equilibrium will be disturbed, because, although CA is greater than CB, the greater weight Q is suspended from A, and the smaller from B, contrary to the condition of equilibrium.

If, therefore, on reversing the weights, we find that the beam still continues horizontal, we may be sure that the arms are equal; but if the beam inclines, we may be sure that the arms are unequal.

False balances are often employed for fraudulent purposes; the arms are made unequal, and to prevent the effect of this being noticed, the scale-pan suspended from the shorter arm is made heavier than that suspended from the longer, so that the unequal weights of the scale-pans just correspond to the unequal lengths of the arms, and balance each other. In this way the beam rests horizontally when no weights are put in the scale-pans. But the substance to be weighed is always put in the pan suspended from the longer arm; therefore it is always lighter than the weights by which it is weighed, and in this manner the buyer receives less of the substance than he ought to do.

Ex. 1.—CA is 7 inches and CB 6 inches; the pan suspended from B weighs one-sixth more than that suspended from A, in which case the beam will rest horizontally when the pans are empty: the weight of the substance which is put in the pan suspended from A, is apparently fourteen ounces; what is its real weight?

Here the scale-pans balance each other, and we

need not consider them; also, P is the substance to be weighed, and Q is the amount of the weights by which P is balanced: wherefore Q is 14, and we have,

$$7 : 6 :: 14 : P;$$

$$\text{and therefore } P = \frac{6 \times 14}{7} = 12 \text{ ounces.}$$

In this case, then, the buyer gets two ounces less than he fancies.

Ex. 2.—On the same supposition, except that the substance is put in the pan suspended from B; to find the real weight.

$$\text{Here we have, } 7 : 6 :: P : 14,$$

$$\text{and therefore } P = \frac{7 \times 14}{6} = \frac{49}{3} = 16\tfrac{1}{3}.$$

In this case the buyer gains $2\frac{1}{3}$ ounces.

Corollary.—To find the true weight by means of a false balance.

Let P be the real weight of the substance to be weighed, Q the amount of the weights which balance it when it is placed in the pan suspended from A, and Q' the amount when it is placed in the other pan. Then,

$$CA : CB :: Q : P, \text{ and therefore } Q = \frac{P \times CA}{CB}.$$

In like manner,

$$CA : CB :: P : Q', \text{ and therefore } Q' = \frac{P \times CB}{CA}.$$

Wherefore, multiplying together Q and Q', we find,

$$Q . Q' = \frac{P \times CA \times P \times CB}{CB \times CA} = P^2.$$

Hence the square of the *true* weight, namely, P, is equal to the product of the *apparent* weights Q and Q', and we may therefore find the *true* weight by multiplying together the two *apparent* weights (the substance being weighed first in one pan and then in the other), and taking the square root of the product.

Ex.—If the substance in one pan appears to weigh 12 ounces, and in the other 3; what is its real weight?

$$\text{Here } P^2 = QQ' = 3 \times 12 = 36,$$

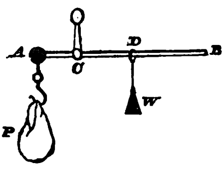
and therefore $P = 6$ ounces, which is the true weight required.

Proposition XLI.

To explain the method of graduating the steelyard, or Roman balance.

This kind of balance, which is commonly used in weighing meat, consists of a beam or lever, AB (fig. 167), suspended by means of a knife edge C, which forms the fulcrum. The substance P to be weighed is suspended from A by means of a hook, and it is balanced by a weight W, which hangs by a ring D, so that it may be moved along CB, and act at any point of it that we please. In this kind of balance, substances are weighed not by being balanced by different amounts of weight, but by shifting the invariable weight W along the arm CB, until P is balanced. The weight of P is therefore determined by the

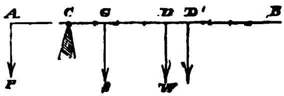
Fig. 167.



position of the ring D on the arm CB, and therefore CB must be marked or graduated in the proper manner, to show the different positions of the ring corresponding to different weights suspended by the hook from A.

To show how CB is to be graduated, let any substance whose weight is P be suspended by the hook, and let the ring be shifted to the proper position on CB, so that the steelyard may be balanced in a horizontal position: then if S be the whole weight of the steelyard (including the hook), and if G be the centre of gravity, the forces which act are W at D, S at G, and P at A. Wherefore, by the Principle of the Lever, we have,

Fig. 168.



$$P \times CA = S \times CG + W \times CD.$$

Again, when a substance weighing $P + 1$ is suspended by the hook, let D' be the point to which W must be shifted; then, as before, we have,

$$(P + 1)\, CA = S \times CG + W \times CD'.$$

Wherefore, by subtraction, we find, observing that $S \times CG$ strikes out,

$$(P + 1) \times CA - P \times CA = W \times CD' - W \times CD,$$

$$\text{or } CA = W(CD' - CD) = W \times DD'.$$

$$\text{Wherefore, } DD' = \frac{CA}{W}.$$

Whence it follows, that when 1 is added to P, the distance CD must be increased by the quantity $\frac{CA}{W}$.

It appears, then, that for every increase of one unit in the weight of the substance P, the ring D must be moved a further distance, equal to $\frac{CA}{W}$, along the arm CB, in order to keep the steelyard in its horizontal position. Thus, if we take inches and ounces for our units, and if CA be *one inch* and W *two ounces*; then $\frac{CA}{W}$ will be $\frac{1}{2}$, that is, DD' is *half an inch*. Wherefore, for every ounce that we add to the weight of P, the distance of the weight W from C must be increased half an inch, in order to keep the steelyard in its horizontal position.

Hence we may graduate the arm CB in the following manner:—First, suspend some known weight from A by the hook, say, for instance, 16 ounces, and move the ring D along the arm CB, until the steelyard is balanced in a horizontal position; mark the position of the ring by scratching a line or graduation on the arm CB, and number it 16. Then suspend 17 ounces, move the ring till the steelyard is balanced horizontally, mark the position of the ring by scratching another line or graduation upon the arm, and number it 17. The distance between these two graduations is equal to $\frac{CA}{W}$, as is evident from what we have proved above, and in this manner we find the exact value of the distance $\frac{CA}{W}$ without knowing CA or W. We have then only to mark a series of equidistant graduations along the arm on each side of the two just found, and

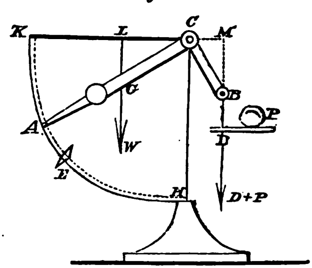
number those on the right, 18, 19, 20, &c. in order, and those on the left, 15, 14, 13, &c. in order, so that the whole set of graduations shall be at the same distance $\frac{CA}{W}$, as just found, from each other in succession. This being done, it is clear, from what we have proved, that, if the ring be at any particular graduation, say 20, when the steelyard is horizontal, the substance P must weigh 20 ounces.

This kind of balance is not suitable for weighing with any great nicety.

Proposition XLII.

To explain the principle of the Bent Lever Balance.

Fig. 169.



The Bent Lever Balance, as the name indicates, consists of a bent lever, ACB, (fig. 169,) C being the fulcrum. The arm CA has a knob near the end, and is shaped as an index at A, corresponding to which is a graduated circular arc, HAK. The arm CB has a scale-pan D suspended from B, in which the substance P to be weighed is put.

The forces which act on this lever are its weight W acting at its centre of gravity G, and the weights D and P together acting at B. Wherefore, if we draw vertical lines through G and B to meet the

horizontal line through C at L and M, we have, by the Principle of the Lever, denoting the weights of D and P by these letters,

$$D + P : W :: CL : CM.$$

Now, as the index A rises, it is clear that LM increases and CM diminishes; wherefore, by the proportion just obtained, $D + P$ must be greater in proportion to W the higher the index is; so that if we continually increase P, A will continually move upwards along the graduated arc. The graduation to which the index points may therefore be made to indicate the weight of P. We might obtain a formula for graduating the circular arc, but it is of no use practically, as the graduation is always done by actual trial. Weights of 1, 2, 3, 4, &c. ounces are put in succession in the scale-pan, and the points on the arc HAK, opposite which the index rests, corresponding to these different weights, are marked and numbered 1, 2, 3, 4, &c. in order. In this manner the arc is easily graduated.

This kind of balance is adapted for weighing quickly where accuracy is not required, because the index immediately shows on the graduated arc the weight of whatever is put in the scale-pan.

We might use this balance without any graduations on the arc, by having a mark E to move along the circular arc. We have only to mark the position of the index by means of E; then take P out of the scale-pan, and put in weights in place of it until the index points exactly to E. The amount of the weights thus substituted for P is evidently the weight of P.

This may also be done with a common false

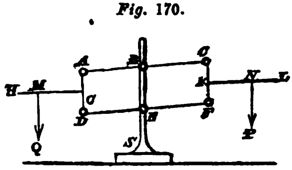
balance.* Put the substance in one scale pan, and balance it by sand placed in the other pan; then take the substance out of the pan, and put weights in place of it, until these weights balance the sand. It is clear that the amount of the weights thus substituted for the substance must be exactly equal to the weight of the substance, whether the arms of the balance be equal or not.

Proposition XLIII.

To explain the principle of Roberval's Balance.

This is the balance generally used in shops for weighing coarse articles, such as coals and the like, and it is extremely convenient for the purpose. The principle of this balance is curious and rather paradoxical, and may be thus explained. Let BES, (fig. 170,) be a stout vertical stand, on which two levers, AC and DF, have their fulcrums, namely, at B and E. These levers are of equal length, B is the middle point of AC, and E the middle point of DF; also they are connected together by two equal rods, AD and CF, having joints at the points A, C, D, and F, so that the levers AC and DF may turn freely about their fulcrums B and E, the figure $ACFD$ being always a parallelogram, and the rods AD and CF being always vertical. Furthermore, GH and KL are arms fixed rigidly and at right angles to the rods AD and CF.

Fig. 170.

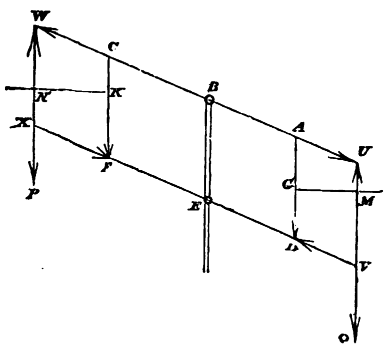


* This method is due to Borda.

Now, the peculiar property of this machine is, that if two equal weights, P and Q, be suspended from the arms GH and KL, they will *always* balance each other, no matter from what points of the arms they may be suspended; that is, whether GM be equal to KN or not, equal weights suspended from M and N will always balance each other.

To prove this, conceive that the lines BA and ED are produced to meet the direction of Q at U and V, as in fig. 171, and suppose $UADV$ to be a rigid parallelogram. Take the line UV to represent Q, and apply, as we may, to the rigid parallelogram, the four forces represented by VU, AU, VD, AD, for these forces balance each other.* Now, of these, the force AU tends to pull the arm AB *directly* away from the fixed point B, and the force VD tends to push the arm DE *directly* against the fixed point E; wherefore these two forces produce no effect, but are destroyed by the reactions of the fixed points B and E. Also the force VU is equal and opposite to Q, and

Fig. 171.



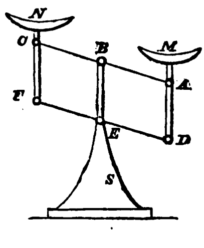
* By Prop. VI. in which it is proved that four such forces balance each other.

therefore the two forces VU and Q destroy each other.

There is then left only the force AD unbalanced; but by a similar process on the other side, which it is not necessary to go through, there is a force CF unbalanced, the other forces, CW, XF, XW, and P, being destroyed, as before: wherefore we have only the two forces AD and CF to consider. Now these, being equal and parallel to each other, and acting on the lever AC at equal distances BA and BC from the fulcrum, balance each other. Thus it appears that P and Q balance each other.

In this demonstration we have not made any assumption that GM is equal to KN, nor is it necessary to do so; whence it appears that P and Q balance each other, provided they be equal, no matter from what points of the arms GH and KL they may be suspended. *Which was to be shown.*

Fig. 172.



In practice, the arms GH and KL are not used; but the scale-pans are fixed to the vertical rods DA and FC, produced upwards, as is shown in fig. 172, at M and N. The scale-pans M and N are of course of equal weight, and when equal weights are put in them they balance each other; but if the weights be unequal, the pan in which the heavier weight is immediately sinks, but is prevented from upsetting the balance by the stand S, which will not allow either of the points D or F to sink more than

an inch or so. This balance is sometimes called the *upsetting balance*, because an inequality in the weights causes the levers $A\,C$ and $D\,F$ to become vertical if allowed, which may be considered as a kind of upset.

From the peculiar principle of this balance, it is not necessary to put the substance to be weighed in the middle of the scale-pan, so that its weight may act directly down the vertical rod which supports the scale-pan. This would not be the case if the scale-pans were attached to the lever $A\,C$ instead of to the vertical rods $D\,A$ and $F\,C$; for, if the balance were constructed in this way, the centres of gravity of the weights put in M and N should be exactly over the points A and C, otherwise the balance could not be depended upon. It would be easy to make a balance of this kind perfectly false, by taking away the lower lever $D\,F$, and fixing the scale-pans to the upper lever $A\,C$ at A and C.

EXAMPLES AND PROBLEMS RELATING TO BALANCES.

Ex. 1.—In the case of the common balance, fig. 163, if P denote the weights, and Q the substance weighed, and if $A\,C = 6$ inches, $C\,B = 6\frac{1}{10}$ inches; how much will a man gain by selling apparently 100 lbs. of tea at 5*s*. per lb., always putting the tea in the scale-pan Q?

Ex. 2.—The buyer requires 50 lbs. weighed in one scale and 50 lbs. in the other; who gains, and how much?

Ex. 3.—The buyer requires 20 lbs. weighed in the scale Q, and 60 lbs. in the scale P; who gains, and how much?

Ex. 4.—The buyer requires 60 lbs. weighed in the scale Q, and 20 lbs. in the scale P; who gains, and how much?

Ex. 5.—The apparent weight of a substance in one scale of a balance is 50 lbs. and in the other scale 60 lbs.; find the true weight, and how much longer one arm is than the other.

Ex. 6.—The apparent weight in one scale is 100 lbs., sold at 2*s.* per lb., and the seller gains 5*s.*; what would be the apparent weight in the other scale?

Ex. 7.—Referring to fig. 165, if $AC = 3$, $CB = 4$, $CG = 1$, $W = 10$, $Q = 100$, and if AB rest at an angle of 45° to the horizon; find P. Do this by measurement or calculation.

Ex. 8.—On the same supposition, except that CG is unknown, and $P = 50$; find CG.

Ex. 9.—On the same supposition as in Ex. 7, except that the inclination of AB is not known, and $P = 50$; find the inclination of AB to the horizon. This may be done by measurement.

Ex. 10.—Graduate the steelyard, supposing that the centre of gravity of the rod AB is at C, that the weight of the hook A is equal to one ounce, and the weight of W, including the ring D, is one ounce; also $AC = 1$ inch. How far is the graduation showing 15 oz. from C?

Ex. 11.—Do the same when the weight of the hook is 5 oz. instead of 1 ounce.

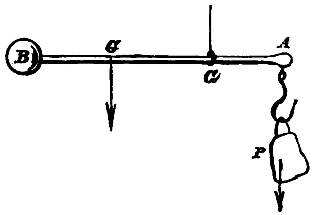
Ex. 12.—On the same supposition, except that AB is a uniform rod, 20 inches long, and weighing 2 oz., graduate the steelyard, and find the distance of the graduation showing 6 oz. from C.

Ex. 13.—In Roberval's balance, if AB be double BC, and DE double EF, show that P is double Q.

Problem XXVI.

To show how a steel yard with a movable fulcrum (the Danish Balance) may be graduated.

Fig. 173.



Let AB, fig. 173, be the steelyard, with a hook at A, for suspending any substance P, and a knob at B, so that the centre of gravity G of the steelyard and hook may lie near B. The fulcrum C is a ring which is held up by a string. Then, if W be the weight of the steelyard, including the hook at A, we have,

$$P : W :: GC : AC :: a - x : x,$$

if we put $AC = x$ and $AG = a$.

Whence we find $Px = W(a - x)$,

$$\text{and therefore } x = \frac{Wa}{P + W}.$$

Suppose, now, that $W = 1$ oz. and $a = 10$ inches, and let us put successively for P the values 1 oz., 2 oz., 3 oz., &c.; then the corresponding values of x will be, in inches,

$$\frac{10}{2}, \frac{10}{3}, \frac{10}{4}, \frac{10}{5}, \text{ \&c.}$$

We have, therefore, only to mark points or graduations on AB at these respective distances from A, and when the fulcrum C is at any one of these points, the steelyard being balanced horizontally,

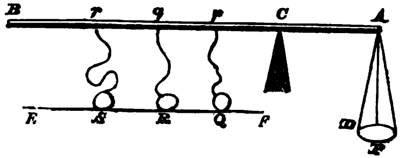
the weight of P will be the corresponding value of P as substituted in the formula; in other words, the successive marks will indicate weights of 1 oz., 2 oz., 3 oz., 4oz., &c. respectively.

Ex.—If AB be a uniform rod, without a knob at B, 2 feet long, and weighing 1 oz.; how far is the graduation showing 11 oz. from A?

Problem XXVII

If AB, fig. 174, be a lever, C the fulcrum, C being also the centre of gravity of the lever, including

Fig. 174.



a scale-pan suspended from A; and if p, q, r, &c. be points of the arm BC to which threads are fastened, connected with weights Q, R, S, &c., which rest on a horizontal plane EF, the string from p being a little shorter than the string from q, which, again, is a little shorter than that from r, and so on; it is required to find what weight P, put in the scale-pan D, will lift one or more of the weights Q, R, S, &c.

This is the principle of Professor Willis's Letter Balance, which is extremely convenient for weighing letters quickly, not so as to determine their actual weight, but for the purpose of ascertaining whether each letter comes up to or exceeds half an ounce, one ounce, two ounces, &c.

If P just lifts Q, we have $P \times AC = Q \times pC$.

If P just lifts Q and R, we have $P \times AC = Q \times pC + R \times qC$.

If P just lifts Q, R, and S, we have $P \times AC = Q \times pC + R \times qC + S \times rC$; and so on.

Let us suppose, for simplicity, that the values of AC, pC, qp, rq, &c. are each one inch; then the different values of P obtained from these equations are,

$$Q, \; Q + 2R, \; Q + 2R + 3S, \text{ \&c.}$$

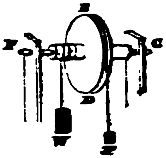
Hence, if we put any unknown weight P into the scale-pan, and find that the string from p is tightened, but Q is not lifted, we know that P is less than Q; if Q is lifted, and the string from q tightened, without R being lifted, we know that P is more than Q, but less than $Q+2R$; if Q and R are lifted, and the string from r tightened, but S not lifted, we know that P is more than $Q+2R$ and less than $Q+2R+3S$; and so on.

Thus, if $Q = 1$ oz., $R = \frac{1}{2}$ oz., $S = \frac{1}{3}$ oz., &c., Q, $Q+2R$, $Q+2R+3S$, &c., will be respectively, 1 oz., 2 oz., 3 oz., and so on: wherefore, when the string from p is tightened, P is under 1 oz.; when the string from q is tightened, P is over 1 oz. and under 2 oz.; when the string from r is tightened, P is over 2 oz. and under 3 oz.; and so on.

WHEELS AND AXLES.

The wheel and axle is nothing but a lever so managed as to admit of the continuous and uniform motion of the power and weight. It consists of

Fig. 175.



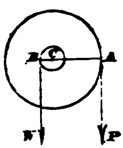
a *wheel*, DE, fig. 175, attac[hed to] an *axle*, FG, which has piv[ots] F and G, round which i[t can] freely move. The *weight* [W is] hung by a string which is [coiled] round and fixed to the ax[le, and] the *power* P which balances [it is] coiled round and fixed to the [wheel.]

Proposition XLIV.

To determine the power which balances a weight by means of a Wheel and Axle.

Let fig. 176 represent the wheel and ax[le] in profile; AP the direction [of the] power, which acts along the [string] hanging from the wheel; B[W the] direction of the weight whic[h acts] along the string hanging fro[m the] axle; and C the common cen[tre of] the wheel and axle, and that [round] which they turn. AC is the radius of the [wheel,] and BC that of the axle. The wheel an[d axle,] then, is a lever, whose fulcrum is C; the [power] acts at an arm CA, and the weight at an ar[m CB.] Hence the condition of equilibrium is, b[y the] Principle of the Lever,

Fig. 176.



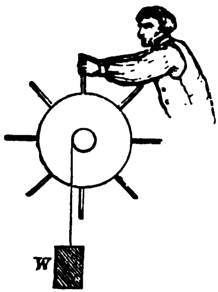
$$P : W :: CB : CA.$$

$$\text{which gives } P = \frac{W \times CB}{CA}.$$

Thus, if $CB = 1$, and $CA = 10$, $P = \frac{1}{10} W$[; that] is, the weight will be balanced by a power [equal] *to one-tenth* of it.

The Capstan.—The capstan is a sort of wheel and axle, only the power is often applied by spokes projecting from the wheel, as is shown in fig. 177. Sometimes, instead of these spokes, there are simply holes in the wheel, into which the man successively inserts a bar, by which he turns the wheel round, there being a contrivance, called a *ratchet wheel*, by which the wheel is not allowed to turn back while the man is taking the bar out of one hole and putting it into the next.

Fig. 177.



Ratchet Wheel.—This contrivance consists of a wheel, MN, fig. 178, attached to the axle and concentric with it, having teeth cut upon it, of the peculiar form shown in the figure. RS is a piece of metal which turns round a pin at R, and rests at S upon the teeth, being pressed down upon them either by its own weight, or by a spring. It is evident that this contrivance will not allow the axle to turn in the direction of the arrow, for the piece of metal RS catches the teeth if the ratchet wheel turns this way. But the axle may turn the other way without obstruction, for the teeth will lift up the extremity S of the piece of metal, and pass under it. RS is called a *click*, from the peculiar noise it makes. The capstan is used on board ship for lifting anchors and other heavy weights. It is often horizontal, as shown in fig. 179.

Fig. 178.

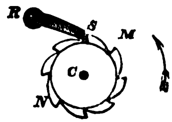
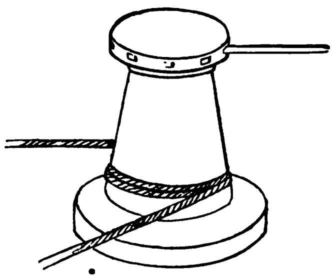


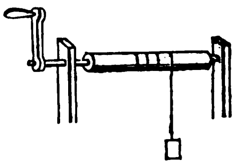
Fig. 179.



A *Winch Handle* is often used instead of the wheel, as is shown in fig. 180. It is so common a contrivance that it is not necessary to say anything about it.

The wheel and axle is evidently a lever capable of raising the weight continuously *to any height*, and it is in this that its superiority over the common lever consists; for the common lever can only lift the weight a short distance, namely, a distance not exceeding twice the length of the arm on which the weight acts.

Fig. 180.

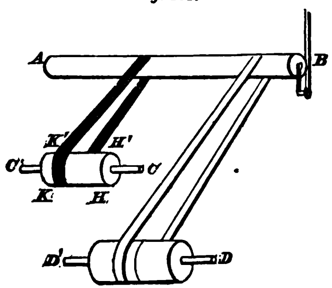


CORDS AND PULLEYS.

The *Cord* is one of the most useful of the mechanical powers, though the simplest. Its primary use is to *transfer power to a distance*, as we know in so many familiar instances. If I cannot reach to a certain body which I wish to move, I tie a cord to it, and by pulling the cord I transmit a force to the body by which I make it move as I require. In the shape of a band passing over axles and setting them in motion, the cord is a most important part of machinery of various kinds. Thus, when a single steam engine

is employed to drive a number of machines in different parts of a building, bands are made to pass from the axle or *shaft* of the steam engine, to the axles of the different machines required to be kept in motion. Thus, in fig. 181, AB represents the great shaft of the steam engine, which is kept in continual rotation by the pressure of the steam acting on the piston, and thence on the crank at B. CC', DD' represent the axles of the various machines required to be turned; and bands passing round the axle AB, and each of these, as shown in the figure, put them in motion.

Fig. 181.



The band which passes over each axle is always sufficiently tight and rough that it may not slip on the axle, but catch hold on it, as it were, and force it to turn. No great degree of tightness or roughness is required, however, for this purpose in most cases.

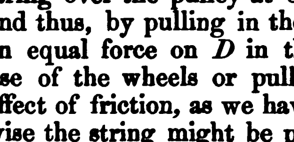
The part HH' of the axle over which the cord passes is called the *drum*; it is firmly attached to the axis and concentric with it. There is generally another drum KK' close by the side of HH', and of the same size as HH'. It is not, however, fixed to the axis like HH', but is capable of turning freely round independently of the axis. The use of this second drum is to stop the motion of the axle CC', or, as it is said, to *put it out of gear*.

This is done by pushing the band *off* the drum HH' *on to* the drum KK'; the band will then act upon the latter drum instead of the former, and since the latter drum turns independently of the axle CC', the motion of CC' is thus arrested, or, to speak more correctly, CC' is no longer moved by the steam engine.

The axle CC' is put *into gear* again by pushing the band back upon the drum HH', which being done, the steam engine immediately begins to turn CC' again, inasmuch as HH' is fixed to CC', so that one cannot turn round without the other doing so also.

Changing the Direction of Forces.—The *cord* is also of great use in changing the direction of forces, which it does by being made to pass over fixed wheels or pulleys. Thus, if by exerting a horizontal pull at A, fig. 182, I wish to pull a body D towards C, all I have to do is to fix a wheel or pulley at C, and another at B, and pass the string over the pulley at C, and under that at B, and thus, by pulling in the direction BA, I exert an equal force on D in the direction DC. The use of the wheels or pulleys is to diminish the effect of friction, as we have stated before; otherwise the string might be passed through rings or holes at B and C.

Fig. 182.



The cord, however, combined with fixed and movable pulleys, is capable of gaining an increase of power, like the lever, and wheel, and axle. Combinations of cords and movable pulleys thus

useful in aiding animal effort. They are
: kinds, which we shall now describe.

PROPOSITION XLV.

'escribe the Movable Pulley, and show its
'cal advantage.

pulley altogether con-
a little wheel AB,
s called the *pulley*, and
f frame CD, which is
:he *block*. The pin or
own at C, about which
eel AB turns, runs
ı the block on each side.
:d runs in a groove cut
he edge of the wheel.
movable pulley the
s not fixed, but has a
t D, or other con-
e, for suspending any
from the block, or
ıg any string to it.
fixed pulley the block
to something, as shown
185.

, let AB, fig. 183, be
ble pulley, having the
W suspended from the
ıt D; the string HA
is fixed at H, passes
the pulley AB, then passes over a fixed
EF, and supports the *power* P. We shall
: that HD, EB, FP, are all vertical lines
first instance.

Fig. 183.

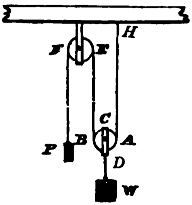


Fig. 184.

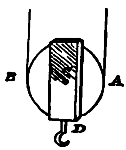
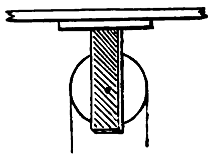


Fig. 185.



The forces which act on the pulley AB are, the tension of the portion AH of the string, the tension of the portion BE, and the weight W, neglecting at present the weight of the pulley itself, or including it in W. Here we have two vertical forces acting at A and B upwards, and a downward vertical force W acting at C half-way between A and B: wherefore, the upward forces must each of them be equal to $\frac{1}{2} W$; that is, the tension on each of the portions AH and EB of the string is $\frac{1}{2} W$. But the tension on EB must be equal to P; for the forces which act on the fixed pulley at E and F, two points equidistant from the centre or fulcrum of the pulley, are the tension on EB and P; wherefore these two forces, since they act at equal arms, must be equal.

It appears, then, that the tension on EB is $\frac{1}{2} W$, and it is also equal to P; wherefore

$$P = \tfrac{1}{2} W.$$

Or, the power required to balance a given weight is half that weight.

Corollary 1.—It is important to observe here, that we assume, in this reasoning, that C is halfway between A and B; otherwise it is not true that the tension on each of the strings is $\frac{1}{2} W$. For instance, if AC were twice CB, the tension on AH would (by the rule for resolving parallel forces) be $\frac{1}{3} W$, and that on BE would be $\frac{2}{3} W$. Hence the above reasoning supposes that C is always half-way between A and B; in other words, it assumes that the pulley AB is truly circular. The same may be said of the pulley EF.

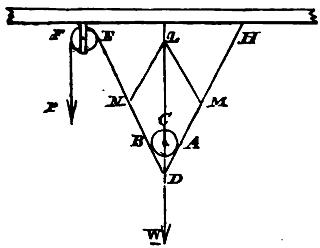
Corollary 2.—Hence, when a pulley is truly

circular, and the two portions of the string which passes round it parallel to each other, we have the following principle, viz.:—

The tension on each portion of the string is equal to half the force which pulls the block.

Strings not parallel.—When the two portions of the string are not parallel, as is represented in fig. 186, produce the lines HA and EB to meet at D; then D is a point on the vertical line CW, because the three forces, namely, W and the tensions on AH and BE, must meet at the same point, and therefore D, the point where the two tensions meet, must be a point on the direction of W.

Fig. 186.



Produce DC to any point L, draw LM parallel to DE, and LN to DH. The two lines DM and DN just touch the circle AB, and therefore the line DC drawn through the centre C must lie half-way between DM and DN, that is, must bisect the angle MDN. It appears, then, that the two strings must be equally inclined to the vertical LD. Also, the lines MD and ND are equal in length.

Let us take the line LD to represent W; then, since we may resolve the force LD into the two forces represented by MD and ND, it follows that MD and ND represent respectively forces which W exercises in the directions of the two portions of the string; in other words, MD and

ND represent the *tensions* on the portions AH and BE respectively. We have then the following construction for finding these tensions.

Draw a vertical line LD, and from D and L draw lines parallel to the directions of the two portions of the string to meet at N, which lines, as we have shown, will be equally inclined to the vertical; then we have the following proportion:

$$\text{tension on each portion of the string} : W :: ND : LD.$$

But we may show as before that the tension on BE is equal to P; wherefore we find,

$$P : W :: ND : LD.$$

By which proportion, when ND and LD are measured, we may find P in terms of W.

Corollary.—It appears that, whether the portions AH and BE of the string be parallel or not, the tension on one must be the same as that on the other, provided of course the pulley is truly circular.

Mathematical solution.—Let the angle which each portion of the string makes with the vertical be θ; then P will be equal to the tension throughout the string, for the reasons just stated, and therefore the forces which keep the pulley AB at rest will be W, P, acting along one portion of the string, and P acting along the other; wherefore, resolving the forces vertically, we have, by the method given in p. 161,

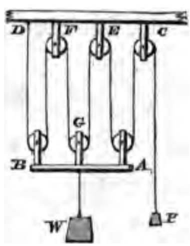
$$2P \cos.\theta - W = 0, \text{ and } \therefore P = \frac{W}{2\cos.\theta}.$$

Proposition XLVI.

To describe the First System of Pulleys, and to find what power will balance a given weight by means of it.

The First System of Pulleys is shown in fig. 187; it consists of a set of fixed pulleys, C, E, F, D; a set of movable pulleys, A, G, B, having their blocks in one piece AB, from which the weight W is suspended; and a single string, which, fastened by one end at D, goes under the pulley B, over F, under G, over E, under A, over C, and then hanging vertically is acted on by the power P. The different parts of this string are all vertical.

Fig. 187.



To find P in terms of W, neglecting the consideration of the weight of the block AB and its pulleys, or including it in W, we have only to observe that the tension on each vertical portion of the string must be the same, by what we have explained in the preceding Proposition; wherefore, since the tension on PC is P, each vertical portion of the string has a tension P acting on it. But W is held up by the tensions of 6 vertical portions of the string. Wherefore a force $6P$ sustains W, and therefore

$$W = 6P, \text{ or } P = \tfrac{1}{6}W.$$

If there were 10 vertical portions of the string holding up W, we should have found $P = \frac{1}{10}W$,

and in general, to find P we have only to divide W by the number of vertical portions of the string which hold up W: or if n be that number,

$$P = \frac{W}{n}.$$

We have here supposed, for the convenience of representation, that the pulleys are arranged along the block AB, as is shown in fig. 187. As this would, however, be an awkward arrangement, the pulleys are so placed as to have a common axis, as is shown in fig. 188, where the fixed pulleys turn in the same block CD, and about the same axis, and the movable pulleys likewise turn in the same block AB, and about the same axis. In this way the whole system is made compact and convenient for use. This is the form of pulley most commonly employed. Its mechanical advantage is immediately known by counting the number of the vertical portions of the string which support the block AB, for W is that number of times greater than P.

Fig. 188.

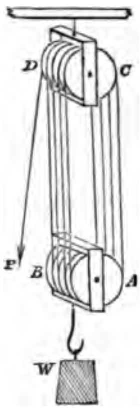
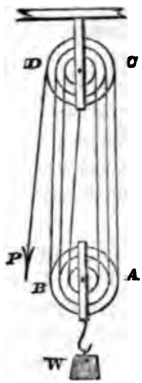


Fig. 189.



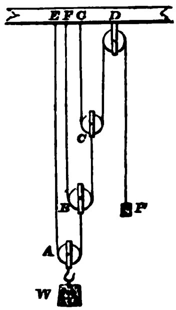
White's Pulleys.—The pulleys in the block AB all turn round with different velocities, and the same may be said of the pulleys in the block CD. They might be made to turn round all with the

velocity, by making them of different sizes, ɔwn in fig. 189, and then the pulleys in the : AB might be all fixed together, or made, ırning the proper grooves of different sizes, ıe piece of wood or metal. In like manner, ulleys in the block CD might be all turned ıe same piece of wood or metal. This, how- does not answer very well in practice.

Proposition XLVII.

describe the Second System of Pulleys, and what power will balance a given weight by s of it.

Fig. 190.



ıe Second System of Pulleys sts of a set of movable ys, A, B, C, fig. 190, and ɔd pulley D; the weight W spended from the block of ı string fixed at E passes r the pulley A and is fastened e block of B; another string at F passes under the pul- ? and is fastened to the block ?; another string fixed at asses under the pulley C, the fixed pulley D, and ing vertically is acted on by the power P. these strings are vertical. We neglect the hts of the pulleys and blocks.

ɔw, by the principle stated in Prop. XLV., tension on each portion of the first of these gs is $\frac{1}{2}W$; wherefore the force acting on the : of B is $\frac{1}{2}W$: wherefore, by the same prin- , the *tension* on each portion of the second

string is *half* of $\frac{1}{2}W$, or $\frac{1}{4}W$; this is therefore the force on the block of C: consequently half of this, or $\frac{1}{8}W$, is the tension on each portion of the third string: but P is equal to the tension on each portion of the third string. It appears, then, that

$$P = \frac{1}{8}W.$$

In like manner, if there were 4 strings we might show that $P = \frac{1}{16}W$, and if there were 5 strings that $P = \frac{1}{32}W$, and so on.

Corollary 1.—In general, if there be n strings, $P = \dfrac{1}{2^n}W$.

Corollary 2.—*To take the weights of the pulleys and blocks into account.*

Let Q denote the weight of each pulley and block; then $W + Q$ is the force on the block of A, and therefore $\frac{1}{2}W + \frac{1}{2}Q$, the tension on each portion of the first string. Wherefore $\frac{1}{2}W + \frac{1}{2}Q + Q$, or $\frac{1}{2}W + \frac{3}{2}Q$ is the force on the block of B. In like manner, the force on the block of C is $\frac{1}{2}(\frac{1}{2}W + \frac{3}{2}Q) + Q$, or $\frac{1}{4}W + \frac{7}{4}Q$. Lastly, P is half of this, that is,

$$P = \frac{1}{8}W + \frac{7}{8}Q.$$

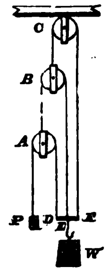
Proposition XLVIII.

To describe the Third System of Pulleys, and find what power balances a given weight by means of it.

This system consists of a set of pulleys, A, B, C, the uppermost one C being a fixed pulley: the weight W is suspended from a piece of wood DE, a string fixed to this piece at D passes over the

pulley A, and hanging vertically is acted on by the power P; another string fixed to the piece at E passes over the pulley B, and is attached to the block of A; another string fixed to the piece at F passes over the pulley C, and is attached to the block of B. The different strings are vertical; we neglect the weights of the pulleys and blocks, but the weight of the piece DF is included in W.

Fig. 191.



Now P is the tension on each portion of the first string, and therefore $2P$ is the force on the block of A; and this is the tension on each portion of the second string: wherefore $4P$ is the force on the block of B, and this is the tension on each portion of the third string. But W is supported by the sum of these three tensions, that is,

$$P + 2P + 4P, \text{ or } 7P.$$

Wherefore $W = 7P$, or $P = \frac{1}{7}W$.

If there were 4 strings we might show that

$$P + 2P + 4P + 8P = W, \text{ or } P = \tfrac{1}{15}W.$$

And so for 5, or any other number of strings.

Corollary 1.—In general, if there be n strings,

$$P + 2P + 4P \;\ldots\; 2^{n-1}P = W;$$

$$\therefore\; P(2^n - 1) = W.$$

Corollary 2.—If Q denote the weight of each pulley and block, we may show, as in the former Proposition, that the tension on each string is, P, $2P + Q$, $4P + 3Q$; therefore,

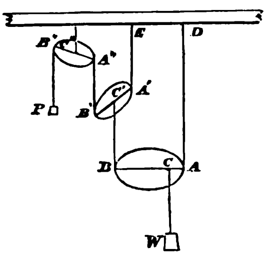
$$7P + 4Q = W.$$

The following Problem will be found instructive, as showing by an example the reason why the tensions on each portion of a string passing round a pulley are equal, and when this is not true.

Problem XXVIII.

To find the condition of equilibrium in the second system of pulleys, supposing that the pulleys are not circular.

Fig. 192.



Let fig. 192 represent such a set of pulleys, AB and $A'B'$ being movable, and $A''B''$ fixed. C, C', and C'' show the axes of the pulleys; and we shall suppose that the lines ACB, $A'C'B'$, $A''C''B''$, are horizontal, and that $CB = 2CA$, $C'B' = 2C'A'$, $C''B'' = 2C''A''$.

Then, by the rules for the equilibrium ot parallel forces, we have the following results:

tension on $DA = \frac{2}{3} W$, tension on $BC' = \frac{1}{3} W$.

In like manner, tension on $B'A'' = \frac{1}{3}$ tension on $BC' = \frac{1}{9} W$.

And in like manner, tension on $B''P$ (which $= P$) $= \frac{1}{3}$ tension on $B'A'' = \frac{1}{18} W$.

Hence we find $P = \frac{1}{18} W$.

If the pulleys were circular we should have found

$$P = \frac{1}{8} W.$$

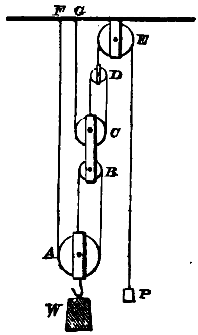
The reason of the difference is, that, in the present case, the tension on AD is not equal to that on BC', nor is that on $A'E$ equal to that on $B'A''$, nor that on $B'A''$ to that on $B''P$; and the reason why these tensions are not respectively equal is, because the points C. C', C'' do not respectively bisect the lines AB, $A'B$, $A''B''$, as they would do if the pulleys were circular.

Problem XXIX.

To find the relation of P to W in the system of pulleys represented in fig. 193.

Here A, B, C, and D, are movable pulleys, and E a fixed pulley. The first string is fixed at F, passes under the pulley A, over the pulley B, and is then fixed to the block of A. The second string is fixed at G, passes under C, over D, and is fixed to the block of C. The third string is fixed to the block of D, and passing over E, is acted on by P. W hangs from the block of A. The blocks of C and B are united in one piece. We neglect the weights of pulleys and blocks.

Fig. 193.



Then, the pulleys being supposed to be circular, the tensions on each of the three vertical portions of the first string are equal to each other; but W is supported by these three tensions: wherefore each of these tensions is $\frac{1}{3}W$. Also, two of these tensions act on the block of C; wherefore the

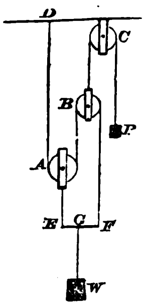
force downwards on the block of C is $\frac{2}{3}W$. In like manner, the tension on each of the three portions of the second string is $\frac{1}{3}$d of $\frac{2}{3}W$, or $\frac{2}{9}W$; wherefore, the downward force on D is $\frac{4}{9}W$, and this is the tension on each portion of the third string. We have, therefore,

$$P = \tfrac{4}{9}W.$$

Problem XXX.

To find the conditions of equilibrium in the system of pulleys represented in fig. 194.

Fig. 194.



Here A and B are movable, and C fixed. The first string is fixed at D, passes under A, over B, and is attached to the bar EF at F; the other extremity E of this bar is suspended by a string from the block of A. W is suspended from the point G of this bar. The second string is fastened to the block of B, passes over C, and is acted on by P.

The tensions on each of the three portions of the first string are equal; the point E is held up by two of these tensions, and F by one of them. Wherefore W is held up by the three tensions, and therefore each tension is $\frac{1}{3}W$. Hence P, which is equal to the upward force on the block of B, and therefore equal to two of these tensions, is $\frac{2}{3}W$. Also, since the force at E is double the force at F, GF must be double GE. Hence the conditions of equilibrium are,

$$P = \tfrac{2}{3}W, \text{ and } EG = \tfrac{1}{3}EF.$$

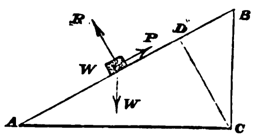
THE INCLINED PLANE.

PROPOSITION XLIX.

A given weight is placed upon a smooth inclined plane; to find what power, acting upwards along the plane, will balance the weight.

Let AB be the inclined plane, fig. 195, AC a horizontal, and BC a vertical line; let W be the weight, and P the power; also, let the reaction of the plane, arising from the pressure of W upon it, be R, which, since the plane is smooth, must be a force acting at right angles to AB. Draw CD at right angles to AB.

Fig. 195.



Then these three forces, W, R, and P, balance each other; but the sides of the triangle BCD are respectively parallel to the directions of these three forces; wherefore, by Proposition VIII., the three forces are proportional to the sides of the triangle, namely, P to DB, W to BC, and R to DC. We have, therefore,

$$P : W :: DB : BC.$$

Which proportion gives P, when DB and DC are known.

Ex.—If DB is one-fourth of BC, $P = \frac{1}{4} W$.

Corollary 1.—The triangle BCD is exactly similar in shape to the triangle ACB, (Euclid, Book VI.) the sides BC, CD, and DB, corresponding to AB, AC, and BC, respectively.

Wherefore, $DB : BC :: BC : AB$, and therefore,

$$P : W :: BC : AB.$$

BC is called the *height*, and AB the *length* of the plane; wherefore it follows, that *the power is to the weight as the height to the length.*

Ex.—If $BC =$ one-eighth of AB, $P = \frac{1}{8} W$.

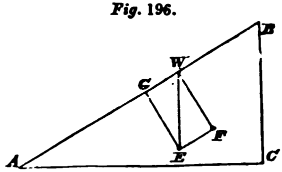
Corollary 2.—R is found by the proportion.

$$R : W :: DC : BC :: AC : AB.$$

That is, the *reaction* (which is equal and opposite to the *pressure exercised by* W upon the plane) *is to the weight as the base* (AC) *is to the length.*

This proposition may be considered somewhat differently by resolving the force W, as follows:

Fig. 196.



Take any vertical line, WE, to represent W; draw WF at right angles to AB, fig. 196, and complete the rectangle $WGEF$. Then the force WE is equivalent to the two forces represented by WF and WG. Of these, the force WG, being perpendicular to the plane AB, is destroyed by the reaction of that plane, which reaction is equal and opposite to WF. If, therefore, we destroy the other force WG, by making a force P equal and opposite to it act on the weight, the weight will be kept at rest on the plane. Hence P is represented by GW, and R the reaction by FW.

In the triangle GWE, WE represents W, GW represents P, and EG represents R; wherefore,

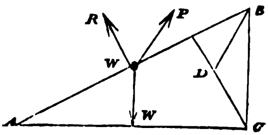
by constructing and measuring this triangle, we may find P and R when W is given. The sides of this triangle are proportional to those of ABC, and therefore we find the same proportions as before.

Proposition L.

If P does not act along the plane; to determine P and R.

Let fig. 197 represent this case, P being inclined at an angle to AB. Draw BD parallel to P, and DC at right angles to the plane, to meet at D. Then, reasoning as in the former proposition, we have,

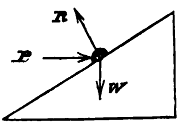
Fig. 197.



$$P : W :: BD : BC,$$
$$\text{and } R : W :: CD : BC.$$

These proportions determine P and R.

Fig. 198.



Examples of the Inclined Plane.

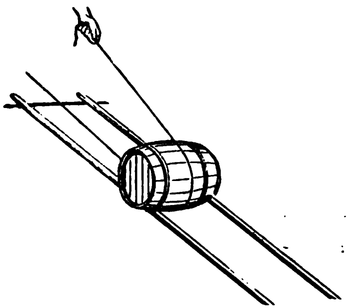
The most familiar example of the use of an inclined plane is a road up a hill. If we suppose W to be a load, fig. 195, and P the force exerted by a horse drawing it up the hill AB, we may see that, if the slope of the hill be small, the horse gains a considerable mechanical advantage; for, suppose the horse had to draw a load of 100 cwt. directly up BC, which we shall suppose to be

100 feet; then the horse must exert a force of 100 cwt. But if the road AB be made at a moderate inclination, say *one foot in ten;* that is, every 10 feet of AB gives a rise vertically of 1 foot; then, since BC is 100 feet, AB will be 1,000 feet; and if the horse pulls the load up this road, exerting a force P, we have,

$$P : 100\text{ cwt.} :: 100 : 1{,}000, \text{ and therefore}$$
$$P = 10\text{ cwt.}$$

Hence the horse, by exerting a force of only 10 cwt., will be able to pull the load up to B.

Fig. 199.



A flight of steps is a familiar instance of the inclined plane. A barrel rolled up to a height on two long poles, fig. 199, is so also. A road winding round a hill, and so gradually leading to the top without any steep ascent, may also be mentioned; and a variety of other well-known instances.

Ex. 1.—P acts along the inclined plane, W is 100 lbs., and the inclination of the plane to the horizon is 30°; find P.

Ex. 2.—Find P on the same supposition, only that the plane is inclined at an angle of 45° to the horizon.

Ex. 3.—Find the same when the inclination is 60°.

Ex. 4.—At what inclination is the plane, when the pressure of the weight upon it is half the weight?

Ex. 5.—If P makes an angle of $30°$ with the plane AB, and AB $30°$ with the horizon; find P when $W = 100$ lbs.

Ex. 6.—On the same supposition, except that the plane AB makes an angle of $45°$ with the horizon; find P and R.

Ex. 7.—On the same supposition, except that AB makes an angle of $60°$ with the horizon; find P and R.

Ex. 8.—In what direction must a power of 75 lbs. act, in order to support a weight of 100 lbs. on a plane inclined at $60°$ to the horizon?

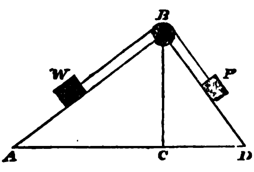
Ex. 9.—Find the same when the inclination is $20°$.

Ex. 10.—The reaction, power, and weight, are all equal to each other; find the inclination of the plane, and the direction of the power.

Problem XXXI.

W and P are two weights resting on two inclined planes AB and BD; a string passing over a pulley at B connects W and P; find the relation between W and P.

Fig. 200.



Let T be the tension on the string; then T is the power that supports W, acting in the direction WB; and T is also the power that

supports P, acting in the direction PB. Wherefore,

$$T : W :: BC : AB, \text{ or } T : BC :: W : AB.$$

Also,

$$T : P :: BC : DB, \text{ or } T : BC :: P : DB.$$

Wherefore,

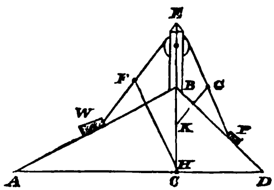
$$W : AB :: P : DB, \text{ or } W : P :: AB : DB.$$

That is, the weights are proportional to the *lengths* of the planes on which they rest.

Problem XXXII.

To determine the same when the pulley over which the string passes is at a certain height above B, as is shown at E in fig. 201.

Fig. 201.



Take any distance EF on one string, and EG equal to it on the other string; draw FH at right angles to AB, and GK at right angles to DB. Then, if T be the tension on the string, W is balanced by its weight W acting parallel to EH, the reaction of the plane AB parallel to HF, and T parallel (or along) to FE. Wherefore,

$$T : W :: EF : EH, \text{ or } T : EF :: W : EH.$$

In like manner, we may show that

$$T : P :: EG : EK, \text{ or } T : EG :: P : EK.$$

Hence, since EF and EG are equal, we find,

$W : EH :: P : EK$, or $W : P :: EH : EK$.

Wherefore, by measuring EH and EK, we may find the proportion of P to W.

Ex. 1.—If the angles BAC, BDC, WEC, and PEC, be respectively 30°, 60°, 15°, and 20°; find the proportion of P to W.

Ex. 2.—If $W = P$, the inclination of $AB = 30°$, and that of $DB = 45°$; find the inclination of EP to the vertical, supposing that of EH to be 45°.

This is done by observing that H and K coincide because P and W are equal: also, the angle FHE is 30°, because FH is perpendicular to AB; the angle GKE in like manner is 45°; also, the angle FEC is given to be 45°. Thus EF is known, and the point G is to be found by describing a circle with E as centre, and EF as radius.

Ex. 3.—If $W = P$, show that there is a certain simple relation in all cases between the angles WEC and PEC.

Ex. 4.—On same supposition as in Ex. 2, except that $W = 2P$; find the inclination of EP to the vertical.

THE WEDGE.

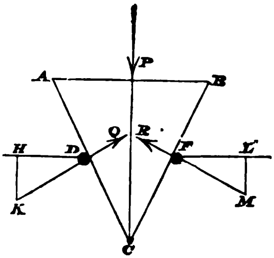
Proposition LI.

To describe the action of the wedge, and show its mechanical effect.

Let ABC, fig. 202, be a block in the shape of a *wedge* or triangle, and let D and F be two obstacles tending to come together, which it is our object to separate, or keep asunder, by driving

the wedge in between them by means of a power P. Let the resistance or reaction of the obstacle D on the side AC of the wedge be Q, and let that of F on BC be R. We shall suppose the surfaces of the wedge to be perfectly smooth, and therefore Q will be at right angles to AC, and R at right angles to BC. We shall also suppose that P acts at right angles to AB.

Fig. 202.



Then the wedge is kept at rest by the three forces P, Q, and R, which are respectively at right angles to the sides of the triangle ABC. Hence, by Cor. 1, Prop. VIII., the forces P, Q, and R must be proportional to AB, AC, and BC, respectively. We have, therefore,

$$Q : P :: AC : AB, \text{ and therefore } Q = \frac{P \times AC}{AB};$$

$$R : P :: BC : AB, \text{ and therefore } R = \frac{P \times BC}{AB}.$$

In this manner we find Q and R, which are the reactions of the obstacles against the sides of the wedge, or, what is the same thing, the pressures which the wedge exerts on the obstacles.

Draw DK at right angles to AC to represent Q, and FM at right angles to BC to represent R; then these two lines show the forces produced immediately on the two obstacles. But we have to consider that the obstacles are not in general

free particles, but are constrained to move in certain directions (see p. 80): let HD be the direction that D is constrained to move in, and LF that in which F is constrained to move; draw KH at right angles to HD, and ML at right angles to LF. Then the force DK is equivalent to two forces, one represented by DH acting along the line DH, and the other acting at right angles to DH and represented by HK. The latter force can produce no effect on D, because D can only move along the line DH, and the former force, namely DH, produces its full effect. Wherefore DH shows the amount of force that is effectively brought into play to move D. In like manner, FL shows the amount of force that is effectively brought into play to move F. Let us represent these two effective forces DH and FL, by Q' and R' respectively; then we have,

$Q' : Q :: HD : DK$, and therefore $Q' = \dfrac{Q \times HD}{DK}$;

and

$R' : R :: FL : FM$, and therefore $R' = \dfrac{R \times FL}{FM}$.

Hence, putting for Q and R their values already found, we have,

$$Q' = P \times \frac{AC \times HD}{AB \times DK};$$

$$R' = P \times \frac{BC \times FL}{AB \times FM}.$$

These are the expressions for the *effective* forces brought into play on the obstacles.

Corollary 1. *Simplified Construction.*—The lengths of DK and FM may be made anything we please; if, therefore, we so draw the figure that DH, FL, and AB, are all equal, the formulæ for Q' and R' become,

$$Q' = P \times \frac{AC}{DK};\ R' = P \times \frac{BC}{FM}.$$

Hence, to find Q' and R' we draw the lines DH and FL in the two directions in which the obstacles are constrained to move, measuring $DH = AB$, and $FL = AB$. We then draw HK and LM at right angles to DH and FL, to meet, at K and M, the perpendiculars to AC and BC drawn from D and F. We then measure AC, DK, BC, and FM, and we so obtain Q' and R' by the formulæ just given.

Corollary 2.—If AC be equal to BC, and if HD and FL be both parallel to AB; then, drawing PC perpendicular to AB, it is evident that the triangle APC is similar to HDK and FML; and therefore we have,

$$HD : DK :: PC : AC,$$

$$\text{and therefore } PC = \frac{AC \times HD}{DK}.$$

$$\text{Hence } Q' = P \times \frac{AC \times HD}{AB \times DK} = P \times \frac{PC}{AB}.$$

And in like manner we may show that

$$R' = P \times \frac{PC}{AB}.$$

In this case, therefore, the effective forces brought into play in separating the obstacles, are got by multiplying P by the fraction $\frac{PC}{AB}$ that is, the fraction formed by dividing the perpendicular from the point C of the wedge upon the base AB by that base.

Examples.—If $PC = 10$, and $AB = 5$; then $Q' = R' = 2P$.

If $PC = 10$, and $AB = 1$; then $Q' = R' = 10P$.

Hence it is evident that the sharper the wedge is, the greater is the power it exerts to separate the obstacles.

The wedge is often employed in practice; in some cases its efficiency depends upon friction, which we do not here take into consideration. A nail is a species of wedge; so also is the edge of any cutting instrument, as a knife. We shall refer to this subject again when we come to speak of friction.

Ex. 1.—Supposing that $AC = BC$, that DH and FL are parallel to AB, and that it requires a force of 1,000 lbs. to be exerted on each obstacle in order to separate them; find what force P acting on the wedge will separate the obstacles, when the angle of the wedge, that is ACB, is $60°$.

Ex. 2.—Find, on the same supposition, what force will separate the obstacles, when the angle of the wedge is $30°$.

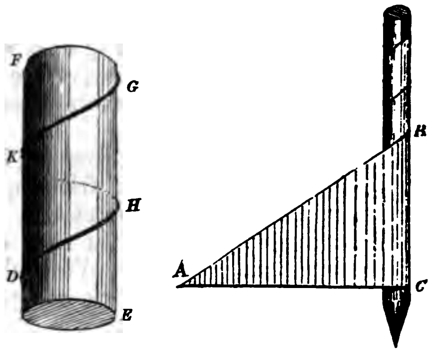
Ex. 3.—Find the same when the angle of the wedge is $10°$.

Ex. 4.—Find the angle of the wedge when the force P required to separate the obstacles is 100 lbs.

THE SCREW.

A *Screw* is a cylinder, *DEGF*, fig. 203, round which a projecting *thread*, as it is called, *DHKG* runs, inclining upwards at a constant angle to the

Fig. 203.

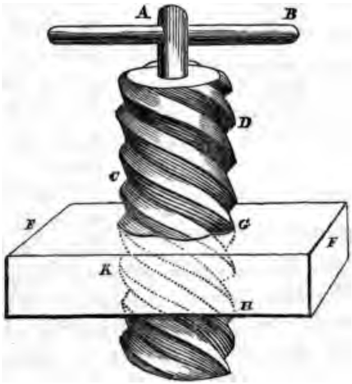


axis or central line of the cylinder, like a road winding up a hill at a constant inclination to the horizon. To get an idea of this, procure a cylinder of wood, a common roller, for instance, or a pencil; cut a piece of paper into the shape of a right-angled triangle *ABC*, and blacken the edge *AB*. Then, fastening the side *BC*, with gum or otherwise, to the cylinder lengthways, wrap the piece of paper round and round the cylinder tightly. The blackened edge will then form a curve running round the cylinder at a constant inclination to the axis of the cylinder. In fact, the blackened edge A B, when wrapped round and

round the cylinder, will be an inclined plane running round the cylinder. This curve represents the *thread of the screw*.

The thread of the screw, *ADC*, is generally cut in the manner shown in fig. 204, and it fits into a hole *GKH*, which has a groove cut in it

Fig. 204.



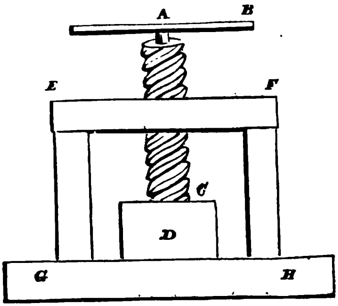
exactly corresponding to the thread of the screw; so that, when the screw is turned round, as, for instance, by the handle *AB*, the thread works accurately in the groove, so that each point of the thread moves down obliquely along the groove, as if it were an inclined plane. In fact, the groove really is a spiral inclined plane, down which the thread runs as the screw is turned round.

Proposition LII.

To show the mechanical effect of the screw.

Let $EFGH$, fig. 205, be a strong frame, in the upper part of which is cut the hole with the groove for the screw to work in. Let AC be the screw, which is supposed to be turned by the handle AB, and so made to move gradually downwards, and squeeze or crush some substance D placed at the bottom GH of the frame.

Fig. 205.

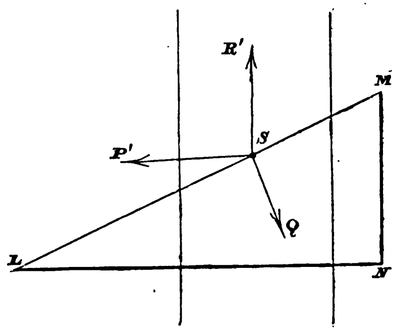


Let P be the power exerted horizontally and at right angles to AB on the handle at B, and R the upward resistance or reaction of the substance D, against the downward pressure exerted by the screw; then these two forces P and R balance each other.

Let S, fig. 206, represent any point of the thread of the screw, which we may regard as a point constrained to move along the inclined plane LM, LM showing the direction in which the groove runs at S; in fact, LMN is the right angled triangle which is spoken of above. The horizontal force P produces a certain amount of horizontal pressure on S; let us represent it by P'; also, the vertical force R produces a certain amount of vertical force on S; let us represent it by R'.

Then S is kept at rest by the three forces P', R', and the reaction Q of the groove at right angles

Fig. 206.



to LM; but LMN is a triangle whose sides LM, MN, and NL, are respectively at right angles to the forces Q, P', and R'. Wherefore we have, by Cor. 1, Prop. VIII.

$$P' : R' :: NM : LN,$$

and therefore $P' = \dfrac{R' \times NM}{LN}$.

Now, if we assume r to represent the radius of the cylinder, the moment of P' about the axis of the cylinder will be $P' \times r$, for r is evidently the arm at which P' acts; also, if we suppose, as we may, that LN is equal to the circumference of the cylinder, in which case L and N will coincide when LMN is wrapped round the cylinder, and MN will evidently be the vertical distance between the threads of the screw; if we make *these assumptions*, putting for brevity d for MN;

and c for LN, we find that the moment of P' is

$$P' \times r, \text{ or } \frac{rd}{c} R'.$$

In like manner, if R'', R''', &c. be the vertical effects produced by R upon the other points of the thread, the moments of the corresponding horizontal effects produced by P on the same points, will be

$$\frac{rd}{c} R'', \frac{rd}{c} R''', \text{ \&c.}$$

And the sum of all these will be

$$\frac{rd}{c} (R' + R'' + R''' + \text{ \&c.})$$

Now this sum must evidently be equal to the moment of the force P which produces all these horizontal forces; also, $R' + R'' + R''' +$ &c. must be equal to R. Wherefore, if we denote AB by a, and therefore the moment of P by Pa, we have,

$$Pa = \frac{rd}{c} (R' + R'' + R''' + \text{ \&c.}) = \frac{rd}{c} R.$$

$$\text{and } \therefore P = \frac{rd}{ac} R.$$

$$\text{or, } P : R :: rd : ac.$$

We thus find P in terms of R, or R in terms of P, when r d a and c are given.

Corollary.—The circumference of a circle is found

by multiplying its diameter by the number 3.14159, which is nearly equal to $\frac{22}{7}$; hence $c = 2r \times 3.14159$, or $\frac{44}{7} r$ nearly. If, therefore, we put this value for c in the expression for P, we find

$$P = \frac{r \times d}{a \times 2r \times 3.14159} \times R = \frac{d}{2\,a \times 3.14159} \times R =$$

$$\frac{7d}{44a} R \text{ nearly.}$$

Hence we have,

$P : R :: d : 2a \times 3.14159$, or $P : R :: 7d : 44a$ nearly.

That is, the Power is to the Resistance as 7 times the vertical distance between the threads to 44 times the arm at which the Power acts, nearly.

Thus for example, if $d = 1$ inch, $a = 35$ inches, then $P = \frac{1}{220} R$, $R = 220\,P$; that is, P exerts a force 220 times greater than itself on the substance D by means of the screw.

The efficiency of the screw, like that of the wedge, depends, however, in many cases upon friction, as we shall show when we come to speak of friction.

Proposition LIII.

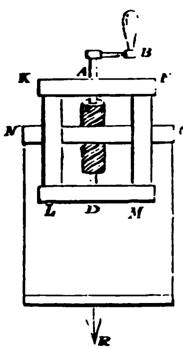
To explain the action, and find the mechanical advantage of the endless screw.

The *endless screw* differs from the common screw

merely in the manner in which it is fixed, which is as follows.

The axis $A\,D$ of the cylinder which forms the screw, fig. 207, is fixed in a frame $KFML$ at A and D; so that, when the handle AB is turned round, the screw does not move upwards or downwards, as in the case of the common screw, but simply turns round its axis $A\,D$. But the groove in which the thread of the screw works is formed in a piece NO, which is capable of moving up and down; whereas, in the common screw, this piece is fixed, being part of the frame. Observe that NO is capable of moving *up* and *down*, but *not of turning round.*

Fig. 207.



We may call the cylinder with the spiral thread running round it an *outside screw*, and the piece in which is cut the corresponding groove, for the thread to work in, we may call an *inside screw.*

Hence the difference between the common screw and the endless screw may be thus stated; the *inside screw is fixed* in the former, but in the latter it is *moveable.*

The resistance R in the endless screw, acts on the piece NO, and we may show exactly as before that

$$P : R :: rd : ac, \text{ or } P = \frac{rd}{ac} R.$$

GENERAL OBSERVATIONS RESPECTING THE MECHANICAL POWERS.

We have now explained the nature of the Mechanical Powers, as commonly enumerated, and calculated the effects which may be produced by them. We must observe, however, that, for the sake of simplicity, we have left out a very important consideration, namely, that the force of friction must necessarily interfere with, and generally diminish the efficiency of the mechanical powers. We shall devote a special Chapter to this consideration, in which we shall show how to estimate the effects produced by friction.

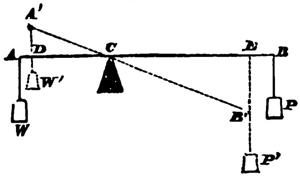
We have also supposed in the preceding Propositions, that the materials which compose the various solid parts of the mechanical powers, are perfectly rigid, hard, and inflexible; also, that the cords, strings, or bands, are perfectly flexible, and inextensible. Now, none of these suppositions are practically true; the materials we assumed to be rigid, are really to a certain extent not so; cords are never perfectly flexible nor inextensible, but have always a certain degree of stiffness, and are always capable of being stretched more or less. Our limits will not allow us to enter upon these considerations, though they are of considerable importance practically; we shall only have space to make a few observations on the extensibility of strings, and the strength and rigidity of materials in a future chapter.

Proposition LIV.

To explain the principle, that what is gained in power, by the Mechanical Powers, is lost in speed.

This is a very important principle in Mechanics, and is one of universal application. No power can be gained by any mechanical contrivance without a corresponding loss of speed. Let us consider the case of the common lever, AB, fig. 208, C being the fulcrum. Suppose the weight W to be elevated to W', by turning the lever round C into the position $A'B'$; then $A'D$ is the space through which the weight W has been elevated, and EB' is the space through which the power P has had to move downward, in order to elevate the weight so much.

Fig. 208.



Now, whatever be the proportion of $A'C$ to CB', the same will be that of $A'D$ to EB'. Also, by the Principle of the Lever, if P balances W, W will be to P in the inverse proportion of $A'C$ to CB'. Wherefore, if W exceeds P any number of times, EB' will exceed $A'D$ the same number of times; that is, the greater W is in proportion to P, the less will the space W is elevated be, in proportion to the space P has to move over to elevate W so much.

Thus, for example, let the arms AC and BC of the lever be respectively 1 foot and 4 feet; then if $A'D$ be 1 inch, EB' will be 4 inches;

also, if W be 100 lbs. P must be at least $\frac{1}{4}$th of W or 25 lbs. in order to elevate W. Here, then, we have a gain of power fourfold; for 25 lbs. is made to elevate 100 lbs.; but there is a proportional loss of speed; for P must move down 4 inches for every one inch that W is elevated. The advantage, therefore, which P gains in power is counterbalanced by an equal disadvantage in speed.

Archimedes said that if he had a firm place outside the earth for a fulcrum, he could move the earth with a lever; let us calculate how long it would take him to move the earth one inch. The weight of the earth is, in round numbers, five thousand millions of millions of millions of tons; suppose Archimedes to exert a power of 1 cwt. to move this weight; then, since the weight exceeds the power one hundred thousand millions of millions of millions of times, the power must move through so many inches in order to move the weight one inch. Now, this number of inches amounts roughly to seventy millions of million circumferences of the earth. Therefore the work Archimedes would have to do would be the same as if he were required to push a body round the earth 70,000,000,000,000 times, all the time exerting a pushing force of 112 lbs.

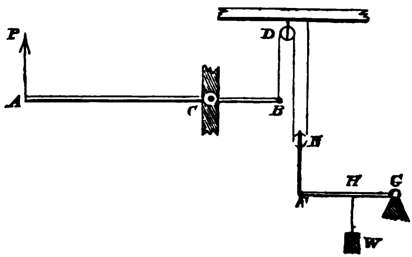
The wheel and axle being a kind of lever, the same principle may be shown to extend to it. It would also be easy to show the same in the case of the other mechanical powers; but, as we shall have to explain the principle in a general way hereafter, when we come to speak of the *Work done by Forces*, we shall not delay to show that it holds for *the other* mechanical powers.

Proposition LV.

To show how to estimate the effect produced by any combination of the mechanical powers.

The mechanical powers may be combined in various ways; for example, let AB be a lever, C its fulcrum, BD a vertical string passing over a fixed pulley D, and under a moveable pulley E,

Fig. 209.



the block of E being connected by a string with the lever GH, whose fulcrum is G. The power P acts at A, and the weight W at H. It is required to find P in terms of W.

Let T represent the tension on the string DB, and T' that on the string EF; then T is the resistance which P has to balance by means of the lever AB; T is also the power which balances the resistance T' by means of the pulley; and T' is the power which balances W by means of the lever GH. We have, therefore, by the preceding propositions,

$$\frac{P}{T}=\frac{CB}{CA}, \quad \frac{T}{T'}=\frac{1}{2}, \quad \frac{T'}{W}=\frac{GH}{GF}.$$

Wherefore, multiplying these together, we find,

$$\frac{P}{T} \times \frac{T}{T'} \times \frac{T'}{W} = \frac{CB}{CA} \times \frac{1}{2} \times \frac{GH}{GF};$$

$$\text{or } \frac{P}{W} = \frac{CB}{CA} \times \frac{1}{2} \times \frac{GH}{GF}.$$

Which expresses what fraction P is of W; for instance, if $CA = 3CB$, and $GF = 4GH$, we find,

$$\frac{P}{W} = \frac{1}{3} \times \frac{1}{2} \times \frac{1}{4} = \frac{1}{24}, \text{ or } P = \frac{1}{24} W.$$

That is, P is $\frac{1}{24}$th of W.

In this combination we may see that the force which is the resistance in the first mechanical power, is the power in the second, and the force which is the resistance in the second is also the power in the third. Now, in general, supposing any number, say four mechanical powers, to be connected in this manner, let P be the power, and R the resistance in the first, R the power, and R' the resistance in the second, R' the power, and R'' the resistance in the third, R'' the power, and W the resistance in the fourth. The fractions $\frac{P}{R}$, $\frac{R}{R'}$, $\frac{R'}{R''}$, $\frac{R''}{W}$, are given by the preceding propositions; let them be respectively, $\frac{m}{n}$, $\frac{m'}{n'}$, $\frac{m''}{n''}$, $\frac{m'''}{n'''}$; then we have,

$$\frac{P}{R} = \frac{m}{n}, \ \frac{R}{R'} = \frac{m'}{n'}, \ \frac{R'}{R''} = \frac{m''}{n''}, \ \frac{R''}{W} = \frac{m'''}{n'''}.$$

Therefore, multiplying all these together, we find,

$$\frac{P}{R} \times \frac{R}{R'} \times \frac{R'}{R''} \times \frac{R''}{W} = \frac{m}{n} \times \frac{m'}{n'} \times \frac{m''}{n''} \times \frac{m'''}{n'''};$$

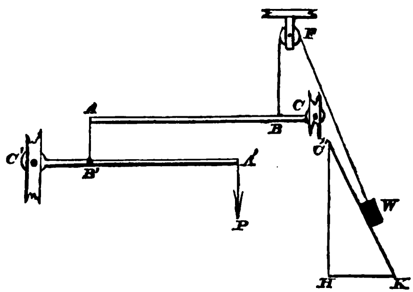
$$\text{or, } \frac{P}{W} = \frac{m}{n} \times \frac{m'}{n'} \times \frac{m''}{n''} \times \frac{m'''}{n'''}.$$

Hence we have the following *Rule* for finding *what fraction the power is of the weight or resistance*, in any machine consisting of a combination of the mechanical powers, such as we have considered.

Find what fraction the power is of the weight or resistance, in each of the mechanical powers, by the preceding propositions; multiply all the fractions together, and the product will be the fraction required.

Example.—One or two more examples of this rule will be useful. Let AC, fig. 210, be a

Fig. 210.



horizontal lever, C the fulcrum; $A'C'$ another horizontal lever, C' its fulcrum; F a fixed pulley;

CK an inclined plane; W the weight, and P the power. A vertical string AB' connects the levers, and another vertical string passing over F draws W up the inclined plane; CH is vertical, and HK horizontal. Let the tension on $AB' = R$, and that on $BF = R'$.

Here we have, by the preceding propositions,

$$\frac{P}{R} = \frac{B'C'}{A'C'}, \quad \frac{R}{R'} = \frac{BC}{AC}, \quad \frac{R'}{W} = \frac{CH}{CK}.$$

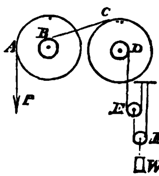
$$\text{Wherefore, } \frac{P}{W} = \frac{B'C'}{A'C'} \times \frac{BC}{AC} \times \frac{CH}{CK}.$$

Thus, if $A'C' = 10B'C'$, $AC = 12BC$, $CK = 2CH$, we find,

$$\frac{P}{W} = \frac{1}{10} \times \frac{1}{12} \times \frac{1}{2}, \text{ or } P = \frac{1}{240}W.$$

Second Example.—In fig. 211, P acts on a wheel at A, the axle of which is connected with another wheel by a string BC, the axle of the second wheel being connected with a pair of pulleys E and F, (forming the *second system,*) by the string DE; W hanging from the block of F. The radius of each wheel is 10, and that of each axle 3; R is the tension on BC, R' that on DE.

Fig. 211.



Here we have, by the preceding propositions,

$$\frac{P}{R} = \frac{3}{10}, \quad \frac{R}{R'} = \frac{3}{10}, \quad \frac{R'}{W} = \frac{1}{4}.$$

And therefore,

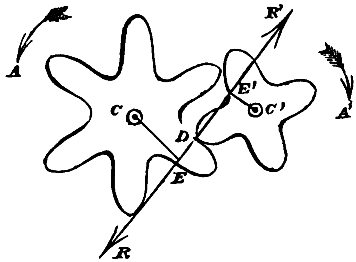
$$\frac{P}{W} = \frac{3}{10} \times \frac{3}{10} \times \frac{1}{4}, \text{ or } P = \frac{9}{400} W.$$

Proposition LVI.

To show the mechanical advantage in a train of cog wheels and pinions.

A *cog wheel* is a wheel with a set of projections, called *teeth* or *cogs*, all round its circumference, as shown in fig. 212. When two wheels of this kind, as C and C', are properly placed with respect to each other, as is shown in the figure, if one of them be turned round, the other will be forced to turn in the opposite direction. Thus, if the wheel C be turned in the direction shown by the arrow A, the wheel C' will be forced to turn in the direction shown by the arrow A'. This action of one wheel on the other is caused by the pressure of the teeth of the one upon those of the other. A tooth of C is always in contact with one of C'.

Fig. 212.



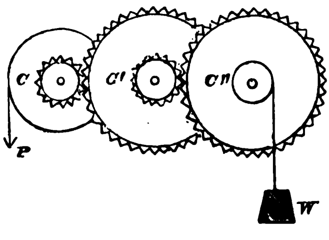
Let D be the point where the tooth of C and that of C', which are in contact, touch each other; and let R' be the force or pressure which the tooth of C exerts on that of C', the latter tooth of course exerting an equal and opposite reaction R upon the former. Draw C E and C' E' perpen-

dicular to the line RR'. Then the moment of the force exerted by the wheel C on the wheel C', about the centre C', is $R \times C'E'$, and the moment of the consequent reaction exerted by the wheel C' on the wheel C, about the centre C, is $R \times CE$.

In general, CE and $C'E'$ are very nearly equal to the radii of the wheels C and C' respectively, and we may assume that CE and $C'E'$ are the radii of the wheels, without any error of consequence. If, then, R be the force of mutual action and reaction between the two wheels, the moment of R about the centre of each wheel is obtained by multiplying R by the radius of that wheel.

Now, let C, C', C'', be a set of wheels and axes furnished with teeth or cogs on their circumferences, and acting upon each other, as is shown in fig. 213; that is, the teeth of the axle of C act on those of the wheel C', and the teeth of the axle C' act on those of the wheel C''. Also, the power P acts on the wheel C, and the weight W on the axle C''. Axles with teeth are called *pinions*.

Fig. 213.



Let R be the force of mutual reaction between the pinion C and the wheel C', and R' that between the pinion C' and the wheel C''; also, let a and b be the respective radii of the wheel and pinion C, a' and b' those of the wheel and pinion C', a'' and b'' those of the wheel and axle C''.

Then the moments of P and R about the centre of C are Pa and Rb, and therefore, by the Principle of the Equality of Moments, we have

$$Pa = Rb, \text{ or } \frac{P}{R} = \frac{b}{a}.$$

And in like manner we find, taking the moments of R and R' about the centre of C',

$$Ra' = R'b', \text{ or } \frac{R}{R'} = \frac{b'}{a'}.$$

And in the same way,

$$R'a'' = Wb'', \text{ or } \frac{R'}{W} = \frac{b''}{a'}.$$

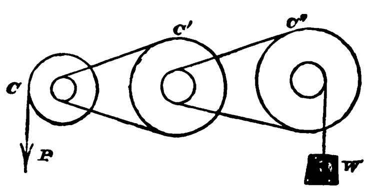
Hence, multiplying all these fractions together, we find

$$\frac{P}{R} \times \frac{R}{R'} \times \frac{R'}{W} = \frac{b}{a} \times \frac{b'}{a'} \times \frac{b''}{a''},$$

$$\text{or, } \frac{P}{W} = \frac{b}{a} \times \frac{b'}{a'} \times \frac{b''}{a''}.$$

Which gives the relation between P and W, and therefore shows the mechanical advantage of the combination of toothed wheels and pinions.

Fig. 214.



Corollary.—If the wheels and axles were connected by bands, as shown in fig. 214, where the axle C acts upon the wheel C', by means of a band instead of teeth, and the axle C' acts upon

the wheel C'' in a similar manner; then it might be shown in exactly the same way, that the same relation holds between P and W as that just obtained in the case of wheels and pinions. In fact, R R' would be the tensions on the bands, instead of the forces of mutual action and reaction between the wheels and pinions.

Example.—In a train of 3 wheels and axles, (or pinions,) the radius of each wheel being 10 times the radius of its axle, to find the mechanical advantage.

Here we have,

$$\frac{P}{W} = \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10}, \text{ or } P = \frac{1}{1000} W.$$

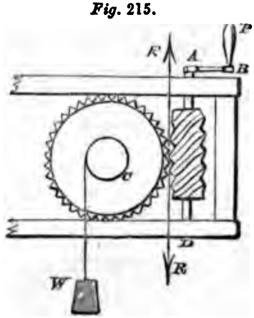
That is, the weight is balanced by a power which is only the thousand part of the weight.

Proposition LVII.

To show the mechanical advantage of the endless screw and cog wheel combined.

AD, fig. 215, is the endless screw, and its thread acts upon the teeth of the wheel C; in fact, the teeth of the wheel C correspond to the *inside* screw in fig. 207, and are moved upwards or downwards by turning the handle AB, exactly in the same manner as the piece NO, fig. 207. The weight W hangs from the axle of C, and the power P acts on the handle AB.

Fig. 215.



Hence, if R be the vertical resistance which the screw has to balance by acting on the teeth; R, or rather a force equal and opposite to R, will be also the power which balances W on the wheel and axle C. If, therefore, a, r, c, and d, denote the same as in Proposition LII. and if a' be the radius of the wheel, and b' that of the axle; we have

$$\frac{P}{R} = \frac{r.d}{a.c}, \text{ and } \frac{R}{W} = \frac{b'}{a'}.$$

$$\text{Wherefore, } \frac{P}{W} = \frac{r.d}{a.c} \times \frac{b'}{a'}.$$

Example.—If the distance (d) between the threads be one inch, the handle AB (a) one foot, and $a' = 10b'$; then, remembering that c is always equal to r multiplied by 3.14159, or $\frac{22}{7}$ nearly, we have

$$\frac{P}{W} = \frac{7 \times 1}{12 \times 22} \times \frac{1}{10}, \text{ or } P = \frac{7}{2640} W.$$

CHAPTER VII.

OF THE FORCE OF FRICTION.

We have already made some general remarks on the subject of *Friction*, and it only remains to state the laws by which the action of this force is regulated, and to show how it may be taken into account, and allowed for in Statical problems.

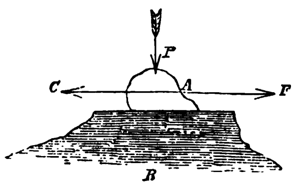
Proposition LVIII.

To state and explain the laws which regulate the action of the Force of Friction.

These laws have been determined by experiment in various ways, and they may be stated as follows:—

All bodies are *rough* more or less, and when placed in contact with each other, they exert a force of resistance to sliding motion, which is called the *Force of Friction*.

Fig. 216.



When two bodies A and B are placed in contact with each other, as shown in fig. 216, if a force be applied to make A slide upon B in the direction $A\,C$, then

a force of friction F is brought into play, in exactly the opposite direction to $A\,C$, whereby the sliding motion of A is prevented altogether, or partially resisted.

The *amount* of this force F, when A is either *actually* sliding upon B, or *just on the point* of sliding, depends upon the pressure P, which presses A directly, that is, *perpendicularly* against B; in fact, F is always proportional to P. For example, if F be 1 oz. when P is 1 lb., then F will be 2 ozs. when P is 2 lbs., 3 ozs. when P is 3 lbs., $\frac{1}{2}$ oz. when P is $\frac{1}{2}$ lb., and so on. Or we may say that F is always the *same fraction* of P; as for instance, in the example just given, F is always $\frac{1}{16}$th of P.

When A is not sliding, nor upon the point of sliding, the force of friction is *indeterminate*, that is, it may be of any magnitude, not, however, exceeding a certain limit, that limit being the amount of the force when A is just on the point of sliding, or actually sliding. In fact, the force of friction, when A is neither actually, nor on the point of sliding, depends upon the force C which pushes A in the direction $A\,C$, and not upon the pressure P, inasmuch as the force of friction is that which prevents the force C from taking effect, and therefore the force of friction must be just equal and opposite to C.

It has been found, however, that when two bodies with perfectly flat surfaces have been some time in contact with each other, they stick together more or less, and a greater amount of friction than that which is in action when sliding is actually taking place, is thus produced. A jar or shake, however, immediately destroys this

tendency to adhere, and then the friction becomes a force of the same magnitude as that which acts when sliding is taking place. It is on this account that the distinction between "*friction of motion*," and "*friction of rest*," has been made, the former denoting the amount of friction in action when the surfaces in contact are sliding one over the other, the latter the amount of friction when they are not sliding. We shall always suppose the friction, in all the cases that follow, to be that of *motion.* The friction of rest is a very variable force, depending upon the time during which the surfaces have been left in contact with each other, and other circumstances. When the bodies are sufficiently tapped or shaken, the friction of rest becomes equal to the friction of motion, supposing of course that there is a sufficient force (C) acting to cause A to be just on the point of sliding.

Coefficient of Friction.—When the body A is actually, or on the point of sliding, the force of friction F is, as we have stated, always *a certain fraction of* the pressure P; that fraction is called the *Coefficient of Friction,* because it is the coefficient by which the pressure P must be multiplied to give the amount of the force of friction F. Thus, if F be always $\frac{1}{16}$th of P, $F = \frac{1}{16} \times P$, and $\frac{1}{16}$ is the coefficient of friction; or if F be always $\frac{1}{3}$d of P, $F = \frac{1}{3} \times P$, and $\frac{1}{3}$ is the coefficient of friction, and so in other cases.

The coefficient of friction is different for different substances, depending upon the degree of roughness or smoothness of the surfaces in contact, upon the *grain* or texture of the substances, *upon the nature of* the unguent or grease which

is often interposed between the surfaces to diminish the friction, and upon various other peculiarities of the substances in contact. The following is a Table exhibiting the amount of the coefficient of friction in a few cases.

Table showing the Coefficient of Friction for certain substances.

N. B.—The fibres of the substance (if it have fibres), are supposed to be parallel to the direction of motion.

Substances.	Coefficient of Friction.
Oak upon oak	.48, or about $\frac{1}{2}$
Wrought iron upon oak	.62 „ „ $\frac{2}{3}$
Cast iron upon oak	.5 „ „ $\frac{1}{2}$
Wrought iron upon wrought iron . . .	.14 „ „ $\frac{1}{7}$
Cast iron upon cast iron	.15 „ „ $\frac{1}{7}$
Brass upon brass	.2 „ „ $\frac{1}{5}$
Oak upon oak, when surfaces were greased and wiped	.11 „ „ $\frac{1}{9}$
Brass upon brass, when surfaces were greased and wiped	.13 „ „ $\frac{1}{8}$

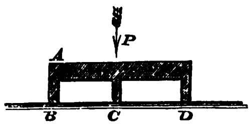
The use of this Table is manifest; thus, if A and B be both of cast iron, the coefficient of friction is $\frac{1}{7}$, and therefore F is $\frac{1}{7}$th of P always. Wherefore, if A be pressed against B by a pressure of 10 lbs., the force of friction, when A is sliding, or on the point of sliding, will be $1\frac{3}{7}$. If the pressure be 30 lbs. the force of friction will be $4\frac{2}{7}$.

PROPOSITION LIX.

To show that the amount of the force of friction is independent of the extent of the surfaces in contact.

Suppose the body A to rest upon a plane BD, at three places, B, C, and D; also, suppose that the coefficient of friction is any particular fraction, say $\frac{1}{2}$, and that the amount of direct pressure of A upon the plane is Q at B, R at C, and S at D. Then the friction at B is $\frac{1}{2}Q$, that at C is $\frac{1}{2}R$, that at D is $\frac{1}{2}S$; wherefore the whole friction is,

Fig. 217.



$$\tfrac{1}{2}Q + \tfrac{1}{2}R + \tfrac{1}{2}S, \text{ or } \tfrac{1}{2}(Q + R + S).$$

Now, if P be the whole pressure which presses A against the plane, it is clear that $P = Q + R + S$; whence it appears that the friction is $\frac{1}{2}P$.

In like manner, if there were 4 points of support instead of 3, and if T were the pressure at the fourth, we might show that the whole friction would be

$$\tfrac{1}{2}(Q + R + S + T), \text{ which} = \tfrac{1}{2}P.$$

Thus, whatever be the number of points of support, it is clear that the force of friction is always the same, namely, one-half of the whole pressure; in other words, the friction does not depend upon the extent of the surfaces in contact, but is always the same fraction of the pressure. The same reasoning would evidently be true if the coefficient of friction were $\frac{1}{3}$ or $\frac{1}{4}$, or of any other value.

N.B.—This reasoning must be restricted to the case *represented* in fig. 217, that is, to friction

between flat surfaces pressed together by a force which does not depend upon the extent of the surfaces in contact: for instance, it does not apply to the case of the friction of the piston in a steam-engine cylinder; or to that of a fluid moving through a pipe.

Laws of Friction briefly stated.

It may be well to state the laws of friction more definitely and briefly as follows:—

1. The force of friction acts always in the opposite direction to that in which the body slides, or tends to slide.

2. The force of friction, when the body is on the point of sliding, is always a certain fraction of the pressure, which fraction is called the coefficient of friction. It is usual to denote this coefficient by the letter μ; and consequently, if the pressure be P, the force of friction will be μP.

3. When the force tending to make the body slide is less than this fraction of the pressure, the body will not slide; when it is greater, the body will slide, provided it be slightly shaken or disturbed, so as to prevent adhesion, and the increase of friction resulting from it.

4. The force of friction is independent of the extent of the surfaces in contact, under the circumstances supposed in Prop. LIX.

5. We may also state, that, if two bodies A and B be in contact, A exercises upon B a force of friction equal and opposite to that which B exercises upon A.

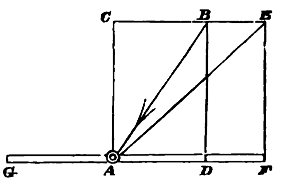
6. Lastly, it is found that friction of motion is the same, whether the motion be quick or slow.

PROPOSITION LX.

To determine within what limits, as regards direction, the resistance of a rough surface acts.

Let A be a body placed upon the surface GF; let us suppose the coefficient of friction to be any particular value, say $\frac{4}{5}$; and let BA represent a force pressing the body A against the plane *obliquely*. Draw AC and BD at right angles to GF, and CE through B parallel to GF; make CE equal to $\frac{4}{5}$ds of CA, draw EF at right angles to GF, and join E and A.

Fig. 218.



Now, the force BA is equivalent to the two forces represented by CA and DA; in other words, the oblique force BA produces a direct or perpendicular pressure CA, and a force DA tending to make A slide along the surface. Now the force of friction arising from the pressure CA would be equal to $\frac{4}{5}CA$, if A were on the point of sliding; but CE, which was made $\frac{4}{5}$ds of CA, is equal to AF, and AD is less than AF; wherefore, the force DA tending to make A slide, is less than $\frac{4}{5}$ds of the pressure, and therefore the body will not slide.

If the force BA acted along the line EA, in which case D and F would coincide, the force DA would be just equal to ^{4}ds of the pressure, and therefore A would be on the point of sliding. And if BA fell on the other side of EA, D would lie *beyond* F, and then the force DA would be

greater than $\frac{2}{3}$ds of the pressure, and therefore A would slide.

Hence the line EA shows the greatest inclination which the force BA may have to the perpendicular CA, without making the body slide; if BA be more inclined to the perpendicular than EA, the body will slide, if less inclined, the body will not slide.

It appears, then, that the surface will resist and prevent the effect of an oblique pressure, such as BA, provided the angle which BA makes with the perpendicular does not exceed the angle EAC. The angle EAC may therefore be called the *angle of resistance.* In the present case this angle is found by making $CE = \frac{2}{3} CA$, and in general, to find the limiting angle of resistance, we must make $CE = \mu \times CA$, μ being the coefficient of friction. Of course, if the surface prevents the effect of the force BA, it must exercise a resistance equal and opposite to BA. When the body is on the point of sliding, therefore, the surface exercises a force of resistance which is inclined at an angle equal to the angle of resistance to the perpendicular.

The angle of resistance is the angle A of a right-angled triangle ACE, in which $CE = \mu CA$, or $\frac{CE}{CA} = \mu$. Those who have begun Trigonometry will see here that the angle of resistance is that angle whose tangent is equal to the coefficient of friction.

Table of the values of the angle of resistance in various cases.

See the Table in p. 326.

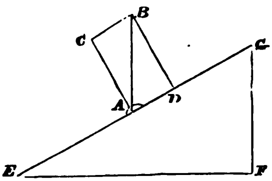
Substances.	Angle of resistance.
Oak upon oak	26°
Wrought iron upon oak	32°
Cast iron upon oak	26°
Wrought iron upon wrought iron	8°
Cast iron upon cast iron	9°
Brass upon brass	11°
Oak upon oak, surfaces greased and wiped . .	6°
Brass upon brass, surfaces greased and wiped .	8°

Proposition LXI.

A body is placed on a plane which is gradually inclined to the horizon; to determine at what inclination of the plane the body begins to slip.

Let EG be the inclined plane, EF horizontal, FG vertical; let A be the body placed upon the inclined plane, and let us suppose it to be just on the point of sliding down the plane: draw BA vertically to represent the weight of A, CA and BD at right angles, and CB parallel to EG.

Fig. 219.



Then the force BA is equivalent to the forces represented by CA and DA, CA at right angles to the plane, and DA along the plane; CA being

the direct pressure against the plane, and DA the force tending to make the body slide down the plane. Since the body does not actually move, the force of friction must be equal and opposite to DA, and is therefore represented by AD. But, since the body is on the point of sliding, the force of friction is equal to the pressure multiplied by the coefficient of friction; that is, AD is equal to AC multiplied by the coefficient of friction, and therefore

$$\frac{AD}{AC} = \text{the coefficient of friction, or } \mu.$$

Now it is easy to show that the triangle ABD is similar to the triangle EFG,* and therefore

$$AD : BD \text{ (or } AC) :: FG : EF;$$

$$\text{whence } \frac{AD}{AC} = \frac{FG}{EF}.$$

$$\text{Therefore } \frac{FG}{EF} = \mu, \text{ or } FG = \mu \times EF.$$

Hence, when the plane is elevated so much, that its height FG is equal to its base EF multiplied by the coefficient of friction, the body is on the point of sliding, and therefore any greater inclination will make it actually slide.

We may obtain this result more simply by considering that the friction is the power which balances the weight of the body; and therefore, by the properties of the inclined plane, as a

* DA is parallel to EF, therefore $\angle BAD = \angle GEF$; also $\angle ADB$ and $\angle EFG$ are both right angles; hence the remaining angles ABD and EGF are also equal, and therefore the triangles are similar.

nechanical power, if we take the length of the)lane to represent the weight, the height will 'epresent the power, that is, the force of friction, ınd the base the direct pressure. Wherefore the ıeight must be the same fraction of the base that :he friction is of the pressure, supposing the body :o be on the point of sliding.

But the best way to consider this proposition is)y reference to the angle of resistance. BA is :he whole oblique force which presses the body A ıgainst the plane EG, and CA is perpendicular to EG; wherefore, since A is supposed to be on the ›oint of sliding, the force AB makes an angle vith CA equal to the angle of resistance. But, ince CA is at right angles to EG, and BA to EF, CA and BA make the same angle with each other hat EF and EG do; wherefore, since the angle BAC is equal to the angle of resistance, the angle FEG is so also.

It appears, therefore, that when the body is on he point of sliding, the angle of inclination of the lane to the horizon is equal to the angle of esistance.

This is equivalent to the former result, because, vhen the angle FEG is equal to the angle of esistance, $FG = \mu EF$, as we have before stated.

Experimental method of finding the Coefficient of Friction.

The present proposition affords a simple method f determining the coefficient of friction for any ›air of substances in contact. We have only to nake an inclined plane of one of the substances, ınd place the other upon it, and gradually elevate

the plane until sliding takes place; we have then only to measure the inclination of the plane to the horizon, and the result will be the angle of resistance, from which the coefficient of friction may be immediately deduced, as is manifest from what has been said above. (Observe, the friction is supposed to be that of motion.)

When a variety of experiments of this kind are tried with different substances, of different sizes, and subject to different degrees of pressure, it is found that the angle of resistance, and therefore the coefficient of friction, determined for any pair of substances, is always the same, no matter what may be the extent of the surfaces in contact, as long as the surfaces have the same degree of polish or roughness, and are not affected by any interposed matter, such as dust, grease, water, or the like. If, however, the pressure be excessive, there appears to be some deviation from the laws of friction as above stated. Thus, in launching a ship, it is found to slip at a less elevation of the inclined plane on which it is placed than it ought to do according to the above laws. This, indeed, is easily accounted for, inasmuch as the wood must be compressed, and made harder than it naturally is, by the enormous pressure arising from the weight of the ship, and therefore, probably, the surfaces in contact are made smoother.

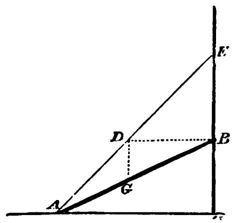
There are many practical difficulties in determining the friction of substances, which we cannot delay to mention. There is, consequently, some discrepancy in the friction tables given by different experimenters.

PROBLEMS SHOWING THE METHOD OF TAKING FRICTION INTO ACCOUNT IN VARIOUS CASES.

PROBLEM XXXIII.

To find how much a beam, AB, fig. 220, *which rests upon a rough horizontal plane AC, and against a smooth vertical plane CE, may be inclined to the vertical without slipping.*

Fig. 220.



Suppose the extremity A to be just on the point of sliding along the horizontal plane CA; then the resistance or reaction of this plane acts obliquely in the direction AE, AE being drawn at an inclination to the vertical equal to the angle of resistance; the reaction of the smooth plane CE acts along BD, which is drawn at right angles to CE, also, the weight of the beam acts vertically through its middle point G. But these three forces keep the beam at rest; wherefore the vertical through G must pass through the point D, as shown in the figure.

Now, because $AG = GB$, and GD is parallel to BE, it follows that $AD = DE$. Hence we have the following construction. Draw any line AE inclined to the vertical at an angle equal to the angle of resistance, that is, make CEA equal to the angle of resistance; from D, the middle point of EA, draw DB horizontally to meet the vertical plane CE at B, and join A and B, then AB shows the required inclination of the beam when

it is on the point of sliding. We have, therefore, only to measure or calculate the angle BAC, and we so determine the inclination required. If the coefficient of friction be given instead of the angle of resistance, we draw EA, by making $AC = \mu EC$.

Ex. 1.—The angle of resistance is 20°, find the inclination of the beam.

Ex. 2.—The angle of resistance is 45°, find the same.

Ex. 3.—The coefficient of friction is ½, find the same.

Ex. 4.—Find the same when the coefficient is 2.

Mathematical Calculation.—Let μ denote the coefficient of friction, then the angle of inclination of the beam to the horizon may be found in terms of μ as follows. Because $AD = DE$, and DB is parallel to AC, it follows that $CB = \frac{1}{2} CE$. But, because AEC is the angle of resistance, $AC = \mu CE$ (Prop. LXI.); wherefore,

$$\frac{CB}{AC} = \frac{\frac{1}{2}CE}{\mu CE} = \frac{1}{2\mu}, \text{ or } CB : AC :: 1 : 2\mu;$$

which proportion determines the angle BAC; in fact, $\frac{CB}{AC}$ is the tangent of the angle BAC.

Thus, in Example 3, $\mu = \frac{1}{2}$, and therefore $CB : AC :: 1 : 1$, or $CB = AC$, and therefore $\angle BAC = 45°$.

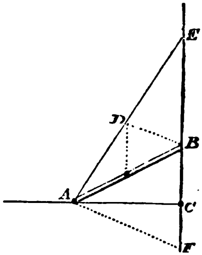
Again, in Example 4, $\frac{CB}{AC} = \frac{1}{4}$, or tan. $BAC = \frac{1}{4}$, which gives BAC by the tables.

PROBLEM XXXIV.

If the vertical plane or wall be also rough, to find the inclination of the beam when it is on the point of sliding.

We use the same letters and construction as before, with this difference, that BD is no longer horizontal, but inclined to the horizon at the angle of resistance, as shown in fig. 221, for, the wall being rough, its resistance does not act horizontally, but at the angle of resistance to the horizon.

Fig. 221.



Observe, BD is inclined *upwards, not downwards*, because, as the beam is on the point of sliding, its extremity B tends to slide downwards, and therefore the friction of the wall acts upwards; consequently, the oblique resistance of the wall, which acts in the direction BD, is inclined upwards, not downwards. For the same reason, the oblique resistance of the horizontal plane on the extremity A, which acts along AE, is inclined *rightwards*, because A tends to slide *leftwards*, if I may coin the words.

The construction then is as follows:—Draw EA as before, making CEA equal to the angle of resistance of the horizontal plane; draw AF inclined downwards to the horizon at an angle CAF, equal to the angle of resistance of the vertical wall, (which may be different from that of the horizontal plane,) bisect EA at D, draw DB

parallel to AF, and join B and A. Then BAC is the angle of inclination of the beam when it is on the point of sliding, and it may be determined by measurement or calculation.

Ex. 1.—Find the inclination of the beam when the angle of resistance is 45° for the horizontal plane, and 30° for the vertical.

Ex. 2.—Find the same when the coefficient of friction is $\frac{1}{2}$ for both.

N.B.—CF is equal to AC multiplied by the coefficient of friction.

Mathematical calculation.—Let μ be the coefficient of friction for the horizontal plane, and μ' that for the vertical; then,

$$AC=\mu.CE,\ CF=\mu'.AC, \text{ and } BF=\tfrac{1}{2}EF.$$

Wherefore, since $BF=BC+CF$, and $EF=EC+CF$, we find,

$$BC+\mu'AC=\tfrac{1}{2}\,(CE+\mu'AC)$$

$$\tfrac{1}{2}\left(\frac{1}{\mu}AC+\mu'AC\right).$$

$$\text{Therefore } BC=\left(\frac{1}{2\mu}+\frac{\mu'}{2}-\mu'\right)AC.$$

And therefore tan. $BAC=$

$$\frac{BC}{AC}=\left(\frac{1}{2\mu}+\frac{\mu'}{2}-\mu'\right)=\tfrac{1}{2}\left(\frac{1}{\mu}-\mu'\right).$$

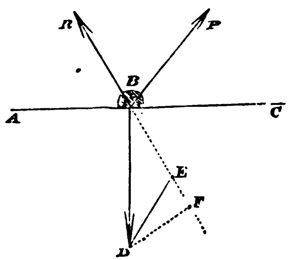
If $\mu'=\frac{1}{\mu}$, we have tan. $BAC=0$, and therefore $BAC=0$; that is, the beam will rest at *any* angle to the horizon, for it will not be on the point of slipping till $\angle BAC=0$.

Problem XXXV.

A body is placed on a rough horizontal plane; to find what force, acting at a certain angle to the horizon, will just make it slide.

Let $A\,C$, fig. 222, be the horizontal plane, B the body, P the force applied to make it slide, P acting at a certain angle $P\,B\,C$ to the horizon. Supposing the body to be just on the point of sliding, then the resistance of the plane, which we represent by R, is not vertical, but acts at the angle of resistance to the vertical. Draw $B\,D$ vertically to represent the weight of the body; produce the line of direction of R backwards to F, and draw $D\,E$ parallel to the direction of P, meeting $B\,F$ at E.

Fig. 222.



Then B is kept at rest by the forces P, W, and R, to the directions of which the sides of the triangle $B\,D\,E$ are respectively parallel. Wherefore, since $B\,D$ has been drawn to represent W, $D\,E$ will represent P. We have, therefore, the following construction for determining the force P, which, acting at a certain angle to the horizon, will just be on the point of moving the body; viz. draw $B\,D$ to represent W, draw $B\,F$, making the angle $D\,B\,F$ equal to the angle of resistance, and draw $D\,E$ parallel to the direction of P; then

DE represents the force P, as required, and may be determined by measurement or calculation.

Ex. 1.—The angle of resistance is 30°, and the weight of the body 100 lbs.; what force P, acting at an inclination of 45° to the horizon, will just be on the point of moving the body?

Ex. 2.—Find the same when P is inclined at an angle of 30° to the horizon.

Ex. 3.—Find the same when P acts horizontally.

Ex. 4.—W being 100 lbs., it is found that a force of 50lbs. acting at an angle of 45° to the horizon, will just move the body; find the angle of resistance.

Mathematical calculation.—Let β be the angle of resistance, and α the angle which P makes with the horizon. Then $\angle DBE = \beta$, $\angle BDE = 90° - \alpha$, and therefore $\angle DEB = 180° - (90° - \alpha) - \beta = 90° - (\beta - \alpha)$; therefore,

$$P : W :: \sin.\beta : \sin.\{90° - (\beta - \alpha)\},$$

$$\text{or, } P = \frac{\sin.\beta}{\cos.(\beta - \alpha)} W.$$

Problem XXXVI.

To determine the best angle of draught in the case just considered.

By the *best angle of draught*, we mean the angle at which P must act, so as to move the body with the greatest ease; that is, so that the least force may be required.

Draw DF, former figure, at right angles to BF. Then DF is the magnitude of P, when P is

parallel to DF, and DE the magnitude of P when P is parallel to DE. But DF is less than DE, in whatever direction DE may be drawn. Therefore, when P acts parallel to DF, P is less than in any other case, and therefore the direction parallel to DF must be the best angle of draught, being that which requires the least amount of force to move the body.

Now, DF being at right angles to BF, and BF making an angle equal to the angle of resistance with the vertical, it follows that DF makes an angle with the horizon equal to the angle of resistance. It appears, therefore, that the best angle of draught is the angle of resistance.

Thus, in Examples 2 and 3, previous Problem, we find that when P is horizontal, it must be greater than when it acts at an angle of 30° to the horizon.

Problem XXXVII.

To find P when the plane AC is inclined at a given angle to the horizon. Also, to find the best angle of draught in the same case.

The construction and reasoning in this case are precisely the same as before, only BF does not make an angle equal to the angle of resistance with the vertical, but with the perpendicular to the plane.

In other words, if β denote the angle of resistance, and a the angle of inclination of the plane to the horizon; BF must be drawn, making the angle $DBF = a + \beta$.

The best angle of draught in this case is β; that is, P makes an angle β with the plane AC,

when the least amount of force is required to move the body.

Ex. 1.—Find P when $a=30°$, and $\beta=30°$.

Ex. 2.—Find P when $a=30°$, and $\beta=45°$.

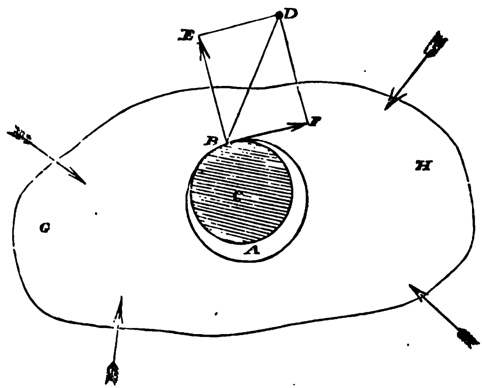
FRICTION OF AN AXIS.

Proposition LXII.

When a body having a cylindrical axis is acted on by given forces, to estimate the effect of friction in retarding the motion of the body about the axis.

Let GH represent the body, having a circular hole through which the axis, shown by the shaded

Fig. 223.



circle BCA, runs, which axis is supposed to be fixed. Let B be the point where the circular hole rests on the axis. Where this point B is, depends upon the manner in which the forces press the body against the axis. We may observe that the

circular hole is supposed to be very nearly equal in diameter to the axis, but of course a little larger than it, to allow freedom of motion.

Draw CE through B from C the centre of the axis; take EB to represent the direct pressure of the body against the axis; and therefore BE represents the consequent reaction of the axis on the body. Draw BF in the contrary direction to that in which the point B of the body tends to move, which is of course at right angles to CE, and take $BF = \mu BE$ (μ being the coefficient of friction), to represent the friction, B being supposed to be on the point of sliding. Then, if we complete the rectangle $BFDE$, the diagonal BD will be the resultant of the direct reaction BE, and the friction BF; and therefore BD represents the total oblique resistance of the axis to the forces which press the body against the axis.

Hence the force BD, and the forces which press the body against the axis, must balance each other; and therefore BD must be equal and opposite to the resultant of the forces which press the body against the axis; in other words, that resultant must be represented by the line DB.

We may here observe, that we may suppose all the forces which keep a body at rest, to act at the same point, whether they really do so or not; see Prop. XXVI. Wherefore, in finding the resultant of the forces which press the body against the axis, we may suppose all these forces to act at B, each of course in, or rather parallel to, its proper direction. DB then, as we have shown, is the resultant of all the forces which press the body against the axis, *each force being supposed to act at B parallel to its proper direction.*

We may also observe, that EBD is the angle of resistance; in fact, DB represents the total oblique pressure which presses the body against the axis, and BE is the perpendicular to the surfaces in contact at the point B; wherefore, since the body is on the point of sliding, the oblique pressure DB must make with the perpendicular BE an angle equal to the angle of resistance.

Hence we have the following simple construction for finding the force of friction BE, when the total oblique pressure DB, which presses the body against the axis, is known, viz.:—

Draw a line BD to represent the total oblique pressure; draw also another line BE, making the angle EBD equal to the angle of resistance, and draw DE at right angles to BE: then DE, which is evidently equal to BF, represents the force of friction.

By a construction of this kind we shall find that DE is always a certain fraction of BD, depending upon the magnitude of the angle of resistance. The following Table exhibits this:—*

Angle of Resistance.	Fraction which the Friction is of the Oblique Pressure.
32° . . .	.53 } about $\frac{1}{2}$, that is, $DF = \frac{1}{2} BD$ nearly.
30° . . .	.5
26° . . .	.44 „ $\frac{3}{7}$
11° . . .	.19 „ $\frac{1}{5}$
9° . . .	.16 „ $\frac{1}{6}$
8° . . .	.14 „ $\frac{1}{7}$
6° . . .	.1 „ $\frac{1}{10}$

* See the Tables in pp. 326 and 331.

To find the effect of the friction in retarding the motion of the body about the axis, we must, of course, take the moment of the force of friction BE, determined as we have just shown, about the centre C of the axis; that is, we must multiply BE by the radius CB. Hence we have the following Rule for estimating the moment or effect of friction in retarding the motion of the body about the axis, namely:—

Multiply the resultant of all the forces which press the body against the axis (the forces being supposed to act at B, each in its proper direction,) by the fraction given in the Table, and by the radius of the axis; and the result will be the moment required.

Mathematical calculation.—We may by a simple calculation, only requiring the 47th Prop. of Euclid, Book I., find what fraction the friction is of the oblique pressure in terms of the coefficient of friction μ; for we have,

$$DE = BF = \mu BE, \text{ and } BE^2 + DE^2 = BD^2.$$

Wherefore, putting μBE for DE, we have,

$$(1+\mu^2)\,BE^2 = BD^2, \text{ or } BE = \frac{BD}{\sqrt{1+\mu^2}};$$

$$\text{and therefore } BF = \mu BE = \frac{\mu}{\sqrt{1+\mu^2}}\,BD.$$

Hence $\dfrac{\mu}{\sqrt{1+\mu^2}}$ is the fraction which the force of friction is of the oblique pressure.

For brevity we shall denote this fraction by the letter λ. The values of λ then are given in the table, which, compared with the table in page

331, will show what λ is for substances of different kinds.

Hence, if R denote the resultant of all the forces which press the body against the axis, and r the radius of the axis, *the retarding moment of the force of friction is*

$$\lambda . R . r, \text{ or } \frac{\mu}{\sqrt{1+\mu^2}} . R . r.$$

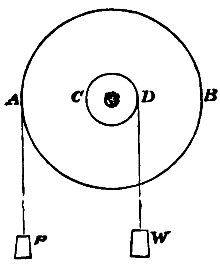
This is a most important rule in Practical Mechanics, and we shall now apply it to one of the mechanical powers, namely, the wheel and axle.

Proposition LXIII.

To find the effect of friction in the case of the wheel and axle.

Fig. 224 represents the wheel and axle with the power P and the weight W; AB being the wheel, CD the *axle*, the inner shaded circle representing the *axis*. Observe, we make a distinction between *axle* and *axis*, the former being that round which the cord DW is coiled, while the latter is the little cylindrical pivot round which the wheel and axle as one body revolve.

Fig. 224.



Let a be the radius of the wheel, b that of the axle, and r that of the axis.

In this case P and W are the forces which press the body against the axis, to which we ought to

add the weight of the wheel and axle itself, which call U; but, for simplicity, we shall not take U into account at first. The resultant of P and W, supposed to act at one point, is $P + W$; wherefore the moment of the force of friction is

$$\lambda (P + W) r.$$

We shall suppose the wheel and axle to be just on the point of moving by the preponderance of P, that is, P is just on the point of pulling up W; then, by the Principle of the Equality of Moments, we have,

$$Pa = Wb + \lambda (P + W) r;$$

observing that, since the friction is always a retarding force, its moment in this case must be contrary to that of P. Hence,

$$P (a - \lambda r) = W (b + \lambda r);$$

$$\text{or } P = \frac{b + \lambda r}{a - \lambda r} W,$$

Which shows what fraction P must be of W in order to be just on the point of drawing up W.

Corollary 1.—If the axis were perfectly smooth, λ would be zero, and we should have,

$$P = \frac{b}{a} W.$$

Corollary 2.—The greater r is, the greater will be the fraction $\frac{b + \lambda r}{a - \lambda r}$; for, as r increases, the numerator increases, and the denominator diminishes. *Hence, the* larger the radius of the axis

is, the greater will be the power required tc a given weight; that is, the greater will be loss of power arising from friction.

Thus, if $a = 10$, $b = 1$, $\lambda = \frac{1}{2}$, and $r = \frac{1}{2}$; find $P = \frac{5}{39} W$. But, if $r = \frac{1}{10}$, $P = \frac{21}{199} W$. $\lambda = 0$, that is, if there be no friction, $P = \frac{1}{10}$ i

Corollary 3.—The loss of power from frictic

$$\frac{b + \lambda r}{a - \lambda r} W - \frac{b}{a} W, \text{ or } \frac{\lambda r (a + b)}{a (a - \lambda r)} W.$$

Ex. 1.—If $a = 20$, $b = 2$, $r = 1$, $c = \frac{1}{2}$; find fraction P is of W.

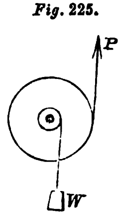
Ex. 2.—If $a = 10$, $b = 2$, $r = 1$; find the p which will raise 100 lbs., the value of μ being

Ex. 3.—If $a = 5$, $b = 2$, $r = 1$, and if the of power arising from friction be $\frac{7}{45} W$; fi and μ.

Proposition LXIV.

To find the effect of friction when the powe the preceding case, pulls vertically upwards, or zontally.

Fig. 225.



If P pulls vertically upwards, represented in fig. 225, the resu of the forces which press the against the axis will be $W - P$, ins of $W + P$; we have, therefore,

$$Pa = Wb + \lambda (W - P) r.$$

And therefore $P = \dfrac{b + \lambda r}{a + \lambda r} W$

We may best compare this with the resu the previous propositions by means of an exa

Let $W = 399$ lbs., $a = 10$, $b = 2$, $\lambda = \frac{1}{2}$, $r = 1$; then, when P pulls downwards, we have,

$$P = \frac{2 + \frac{1}{2}}{10 - \frac{1}{2}} W = \frac{5}{19} 399 = 21 \text{ lbs.}$$

But if P pulls upwards, we have,

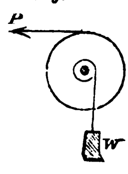
$$P = \frac{2 + \frac{1}{2}}{10 + \frac{1}{2}} W = \frac{5}{21} 399 = 19 \text{ lbs.}$$

Hence we see that a smaller power is required when it acts upwards than when it acts downwards.

The reason of this is obvious, because the friction depends upon, and is proportional to the force which presses the body against the axis, and this force must be less when P and W act in opposite directions, than when they act in the same directions.

Hence it should be always borne in mind, in devising machinery in which there is rotation about axes, that the forces which press each body against its axis should be made to oppose each other as much as possible, so that the total pressure they produce should be as small as possible, and therefore the friction also. This rule is well known to good practical mechanics.

Fig. 226.



If P pulls horizontally, as is represented in fig. 226, the question becomes a little more complicated; for then the force which presses the body against the axis is the resultant of two forces acting at right angles to each other, namely, P and W; and

by Prop. XIII. this resultant is $\sqrt{P^2+W^2}$; we have, therefore,

$$Pa = Wb + \mu r \sqrt{P^2+W^2}.$$

And from this, by the solution of a quadratic equation, we may find P in terms of W*. But a simpler solution may be obtained, which answers perfectly well in all practical cases, by considering that r is always small, and so is μ; for the axis about which the body turns is always made as small and as smooth as possible in order to diminish the effect of friction; the amount of friction brought into play being proportional to μ and r jointly. Hence we may regard $\mu r \sqrt{P^2+W^2}$ as small compared with Wb, and therefore we need not be very accurate in finding the value of $\mu r \sqrt{P^2+W^2}$. Now, since this quantity is small compared with Wb, it follows that $Pa = Wb$ nearly, and therefore $P = \frac{b}{a} W$ nearly. Let us put this value of P in the quantity $\mu r \sqrt{P^2+W^2}$, which, as we have said, we need not determine very accurately; and then the above equation becomes,

$$Pa = Wb + \mu r \sqrt{\frac{b^2}{a^2} W^2 + W^2};$$

$$\text{or } P = \left(\frac{b}{a} + \frac{\mu r}{a} \sqrt{\frac{b^2}{a^2} + 1}\right) W.$$

Which is a tolerably simple expression for P.

* $$(Pa - Wb)^2 = \mu^2 r^2 (P^2 + W^2).$$

$$\therefore P^2 - \frac{2abW}{a^2 - \mu^2 r^2} \cdot P + \frac{b^2 - \mu^2 r^2}{a^2 - \mu^2 r^2} W^2 = 0.$$

Which is a quadratic for finding P.

Corollary 1.—*To find what additional power is required in consequence of the friction.*

If the axis was perfectly smooth, that is, if μ was zero, the value of P would be $\frac{b}{a} W$; call this P'. Now, by the formula for P, we find that the power actually required when μ is not zero, is,

$$\frac{b}{a} W + W \frac{\mu r}{a} \sqrt{\frac{b^2}{a^2} + 1}, \text{ or } P' + P'.\frac{\mu r}{b} \sqrt{\frac{b^2}{a^2} + 1}.$$

Hence the additional power required in consequence of the friction is,

$$P'.\frac{\mu r}{b} \sqrt{\frac{b^2}{a^2} + 1}.$$

Thus, if $a = 100$, $b = 10$, $r = 1$, $\mu = \frac{1}{4}$, we find,

$$\frac{\mu r}{b} \sqrt{\frac{b^2}{a^2} + 1} = \frac{1}{40} \sqrt{\frac{101}{100}} = \frac{1}{40} \text{ nearly.}$$

Hence the power required in this case is $\frac{1}{40}$th more than the power that would be required if the axis were perfectly smooth.

Corollary 2.—We may see, from the formula just obtained for the additional power required in consequence of friction, that the smaller r is, the less that additional power will be in proportion. Hence the importance of making axles as small as possible is manifest.

Proposition LXV.

To obtain a simple practical rule for estimating the additional power required in consequence of friction.

in the case of the wheel and axle, whatever way the power may act.

The greatest pressure will be experienced by the axis when P acts vertically downwards, and the least when P acts vertically upwards; in the former case,

$$Pa = Wb + \mu r (W + P).$$

Or, putting $\frac{b}{a} W$ for P in the small quantity $\mu r (W + P)$, we find,

$$P = \frac{b}{a} W + \frac{\mu r}{a} \left(\frac{b}{a} + 1\right) W.$$

Or, using P' as before,

$$P = P' + \frac{\mu r}{b} \left(\frac{b}{a} + 1\right) P'.$$

Hence the additional power required is

$$\frac{\mu r}{b} \left(1 + \frac{b}{a}\right) P'.$$

Again, in the latter case, where P acts upwards, we find, in the same way, that the additional power required is $\frac{\mu r}{b} \left(1 - \frac{b}{a}\right) P'$. Hence we have the following rule:—

If P' be the power required on the supposition that the axis is perfectly smooth, then the additional power required in consequence of friction will be

$$\text{not less than } \frac{\mu r}{b} \left(1 - \frac{b}{a}\right) P',$$

ınd not greater than $\frac{\mu r}{b}\left(1+\frac{b}{a}\right)P$.

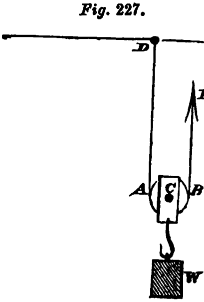
example, if $a=100$, $b=10$, $r=1$, $\mu=\frac{1}{4}$; d that the additional power required is not ıan $\frac{8}{400}P'$, and not greater than $\frac{11}{400}P'$.
s is a very useful way of simplifying a rule, y, by finding not the exact quantity required, rtain limits, which give a tolerably fair idea magnitude.

PROPOSITION LXVI.

ıstimate the effect of Friction in the case of the Moveable Pulley.

A B, fig. 227, be the pulley, *P* the power, e weight; and suppose is just so great as to the point of overcoming l the friction, so that the ıddition to *P* will proıpward motion. Now, ction takes place at the axis *C*, round which lley turns; *W* is manithe total force which this pin against the , and therefore of course ı reacts on the pulley with the same pres- The friction is therefore λW, (see p. 346,) ıe moment of it about the centre of the as fulcrum, is λWr, r being the radius of s. Now the forces which prevent the motion pulley about its axis are the friction, the , and the tension of the string *A D*, (which

Fig. 227.



call T.) P and T act contrary ways, botl arm a, a being the radius of the pulley the friction acts against P, because it te resist the motion which is about to take p the direction of P. We have, therefore, Principle of the Lever, the following equat

$$Pa = Ta + \lambda Wr \; . \; . \; . \; . \; . \quad (1.)$$

Also, if we suppose, as we may do by III. (page 85,) that the pulley and block rigidly connected together, we have

$$P + T = W \; . \; . \; . \; . \; . \; . \quad (2.)$$

Hence, multiplying (2) by a, adding it and cancelling Ta, we find,

$$2Pa = aW + \lambda rW.$$

$$\text{or } P = \tfrac{1}{2}W + \frac{\lambda r}{2a}W \; . \; . \quad (3.)$$

This gives P in terms of W, or, in other it determines the power (P) just suffic begin elevating the weight W.

Corollary 1.—The power required, if th no friction, would be $\frac{1}{2}W$, (see page 282,)* the *additional* power required in consequ friction is,

$$\frac{\lambda r}{2a}W.$$

This additional power is diminished

* This may be easily seen by putting $\lambda = 0$ in the for which amounts to supposing that there is no frictio remark is worth attending to.

diminish r, or increase a, as is manifest from the formula. Hence it is important to make the pulley as large, and the pin or axis as small as convenience and due strength permit.

Corollary 2.—From (1), by substituting $P+T$ for W, we find,

$$Pa = Ta + \lambda\,(P+T)\,r.$$

$$\text{And} \therefore\ P = \frac{a+\lambda r}{a-\lambda r}\,T \ldots (4.)$$

This is an important formula, for it gives a rule for finding how much greater the tension on one side of the string passing over a pulley is, than the tension on the other side; for the friction of the axis will always cause one tension to exceed the other.

Corollary 3.—If we suppose that P is only just sufficient to hold up W, that is, that W is on the point of moving downwards, the friction will tend the opposite way to that supposed in the above investigation. Instead of (1), therefore, we shall have,

$$Pa + \lambda\,Wr = Ta.$$

This will make,

$$P = \tfrac{1}{2}\,W - \frac{\lambda r}{2a}\,W.$$

Hence the condition of equilibrium in the case of the single moveable pulley, taking friction into account, may be thus stated, viz.:—If the power is not less than $\frac{1}{2}\,W - \frac{\lambda r}{2a}\,W$, or not greater than $\frac{1}{2}\,W + \frac{\lambda r}{2a}\,W$, it will balance the weight W.

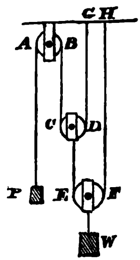
For example, if $W=200$, $\lambda=\frac{1}{2}$, $a=10r$, the value of $\frac{\lambda r}{2a}W$ is 5; therefore, 105 lbs. will just begin to elevate 200 lbs., and 95 lbs will be just sufficient to hold up the 200 lbs. and prevent it from going down.

If $a=2r$, the elevating force will be as much as 125 lbs. and the holding force only 75 lbs. We may see from this what a difference the size of the axis in proportion to that of the pulley makes.

Proposition LXVII.

To estimate the effect of friction in the case of the system of Pulleys represented in the annexed figure.

Fig. 228.



The figure needs no description, the system of pulleys it represents having been treated of in the preceding Chapter. The radius of each pulley is supposed to be a, and that of each axis r. Then, by *Corollary* 2, Proposition LXVI., the tension on the string BC, (supposing P to be on the point of drawing up W,) is

$$\frac{a-\lambda r}{a+\lambda r}P, \text{ or } cP, \text{ putting } \frac{a-\lambda r}{a+\lambda r}=c, \text{ for brevity.}$$

For the same reason, the tension on DG is c times that on BC, which makes tension on $DG=c^2P$. But the tension on the string EC is equal to the sum of the tensions on CB and DG; therefore,

tension on $EC=cP+c^2P=(c+c^2)P.$

Again, as before, the tension on FH is c times that on EC; therefore,

$$\text{tension on } FH = (c^2 + c^3)\, P.$$

Lastly, $W =$ tension on EC + tension on $FH =$

$$(c + c^2)\, P + (c^2 + c^3)\, P = (c + 2c^2 + c^3)\, P.$$

We have therefore,

$$P = \frac{W}{c\,(1 + c)^2}.$$

Which determines the power necessary to begin elevating W.

Example.—Suppose that $\lambda = \frac{1}{8}$, $a = 3\,r$, and therefore $c = \frac{8}{10} = \frac{4}{5}$. Then,

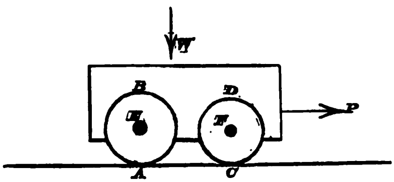
$$P = \frac{W}{\frac{4}{5}\left(1 + \frac{4}{5}\right)^2} = \frac{125}{324}\, W.$$

Proposition LXVIII.

To estimate the effect of friction in the case of carriage wheels.

Fig. 229 represents the carriage; W is its total

Fig. 229.



weight, and P the horizontal power just sufficient to begin *moving it*; AB and CD are the wheels,

E and F their axles. The wheels are supposed to be equal, and their axles also; the radii of the former equal to a, and those of the latter to r. The ground AC is assumed to be perfectly flat and horizontal.

The force which presses the carriage against the two axles E and F, is evidently the resultant of P and W; but P is always small compared with W, as we know by experience; therefore, the resultant of P and W is very nearly the same thing as W, and consequently W may be practically regarded as the force which presses the carriage against its axles. Let W' denote the pressure exerted on E, and W'' that on F; then,

$$W' + W'' = W \quad . \; . \; . \quad (1.)*$$

Now, the friction arising from W' is $\lambda W'$, and this is a force acting on the wheel AB at an arm r, tending to prevent its turning round. The horizontal resistance of the ground at A is the only other force tending to turn AB round E; call this force R, then, since R acts at an arm a, and just balances $\lambda W'$ acting at an arm r, we have,

$$Ra = \lambda W' r, \text{ and } \therefore \; R = \frac{\lambda W' r}{a}.$$

And, in like manner, if R' denote the horizontal resistance of the ground at C, we shall find,

$$R' = \frac{\lambda W'' r}{a}.$$

* $W' + W''$ *might* be much greater than W in a shaky ill-constructed carriage, a case which is not contemplated here.

And therefore,

$$R + R' = \frac{\lambda W' r}{a} + \frac{\lambda W'' r}{a} = \frac{\lambda W r}{a} \text{ by (1.)}$$

Now suppose, as we may, that the wheels become rigidly united to their axes, so that the carriage and wheels become one rigid body; then, since R, R' and P are the only horizontal forces acting, and since P is just on the point of overcoming these two forces, we have,

$$R + R' = P, \text{ and } \therefore\ P = \frac{\lambda W r}{a}.$$

Hence it appears that the force just necessary to begin moving the carriage is,

$$\frac{\lambda W r}{a}.$$

Corollary.—This force is greater the greater $\frac{r}{a}$ is, and therefore, in order to make a carriage as free to move as possible, the wheels should be as large as convenience and stability will allow, and the axles as small as is consistent with due strength.

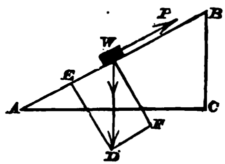
Proposition LXIX.

To estimate the effect of friction in the case of the inclined plane.

Let AB, fig. 230, be the inclined plane, W the weight resting on it, P the power supporting W, μ the coefficient of friction, AC horizontal,

and BC vertical. Draw WD vertically downwards to represent W, WF at right angles to AB, and complete the rectangle $WEDF$.

Fig. 230.



Then WF is the force which presses W perpendicularly against the plane, and therefore μWF is the friction. Also, P acts along the plane upwards, and the force represented by the line WE acts in the contrary direction. Hence, if W is on the point of being moved *up* the inclined plane, in which case the friction will act *down*, we have,

$$P = EW + \mu FW.$$

Now, the triangle ABC is evidently similar to the triangles WDF and WED; therefore,

$$EW : WD :: BC : AB,$$

$$\text{and} \therefore\ EW = WD \times \frac{BC}{AB} = W \times \frac{BC}{AB}.$$

Also, $$FW : WD :: AC : AB,$$

$$\text{and} \therefore\ FW = WD \times \frac{AC}{AB} = W \times \frac{AC}{AB}.$$

$$\text{Hence } P = W\left(\frac{BC}{AB} + \mu\frac{AC}{AB}\right).$$

This is the power necessary just to begin moving W up the plane.

If, however, W be on the point of moving down the plane, the friction will act in the opposite direction, and therefore we find,

$$P = EW - \mu FW.$$

And therefore, as before,

$$P = W\left(\frac{BC}{AB} - \mu\frac{AC}{AB}\right).$$

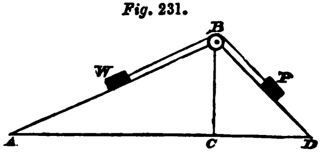
This is the power just necessary to hold W, and prevent its moving down the plane.

The former value of P may be called the *moving power*, and the latter the *holding power*. Any power between the two values will produce equilibrium.

Corollary.—To find the proportion of P to W in the case represented by fig. 231.

The figure represents W and P connected by a string, passing over a pulley B, (supposed to be perfectly smooth,) W rests on the inclined plane AB, P on the inclined plane DB, AD is horizontal, and BC vertical.

Fig. 231.



Let P be a moving power, that is, suppose that P is just on the point of drawing W up the inclined plane AB; let T be the tension of the string, which will be the same on W as on P. Then T is a moving power on W, and a holding power on P, and therefore, by the Proposition, we have,

$$T = W\left(\frac{BC}{AB} + \mu\frac{AC}{AB}\right).$$

$$T = P\left(\frac{BC}{DB} - \mu\frac{DC}{DB}\right).$$

Equating these two values of T, we find,

$$\frac{P}{W} = \frac{DB}{AB} \times \frac{BC + \mu AC}{BC - \mu DC};$$

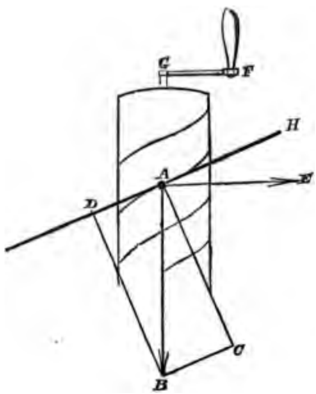
which gives the required proportion.

Proposition LXX.

To estimate the effect of friction in the case of the screw.

Suppose that a power P is employed to turn the handle GF (fig. 232) of the screw, and that it just begins to raise a weight, or overcome a resistance, which we shall represent by W. Let A be any point of the thread of the screw; let w be the portion of W supported at A, and p the force acting horizontally in the direction AE, just sufficient to support w, and overcome the friction resulting from the pressure of w. In fact, we may conceive w to be a weight placed on an inclined plane at A, (see page 304,) and p to be the corresponding *moving* power acting horizontally. AB is drawn vertically downwards to represent w, AC is at right angles to the inclined plane, AE represents p, D A H shows the direction

Fig. 232.



of the inclined plane, BD and EH are perpendicular to DAH.

The perpendicular pressure on the inclined plane arises partly from w and partly from p; DB represents the part due to w, EH the part due to p; therefore, the pressure is altogether $DB+EH$, and consequently the friction is $\mu(DB+EH)$. AH is the force tending to make A move up the plane, and AD that tending to make A move down; also, since A is on the point of moving up the plane, the friction acts down. Hence we have,

$$AH = AD + \mu(DB+EH).$$

Now, let b, h, and l, represent respectively the base, height, and length of the inclined plane, (see page 307); then, since BAD and EAH are similar to the inclined plane, we have,

$$AH : AE \text{ (or } p) :: b : l; \therefore AH = p\frac{b}{l}.$$

$$AD : AB \text{ (or } w) :: h : l; \therefore AD = w\frac{h}{l}.$$

$$DB : AB \text{ (or } w) :: b : l; \therefore DB = w\frac{b}{l}.$$

$$EH : AE \text{ (or } p) :: h : l; \therefore EH = p\frac{h}{l}.$$

Hence, in the equation just obtained, we find,

$$p\frac{b}{l} = w\frac{h}{l} + \mu\left(w\frac{b}{l} + p\frac{h}{l}\right);$$

$$\text{and therefore } p = \frac{h+\mu b}{b-\mu h}\, w.$$

Now, let w', w'', w''', &c. be the several portions of W, supported at the other points of the thread of the screw, and let p', p'', p''', &c. be the corresponding moving powers. We may show then, by the process just gone through, that,

$$p' = \frac{h + \mu b}{b - \mu h} w', \; p'' = \frac{h + \mu b}{b - \mu h} w'' \text{ \&c. \&c.}$$

Hence, by addition, we find,

$$p + p' + p'' + \text{\&c.} = \frac{h + \mu b}{b - \mu h} (w + w' + w'' + \text{\&c.})$$

Now here we have, evidently, $w + w' + w'' +$ &c. $= W$. Also, p, p', p'', &c. are forces acting horizontally at different points of the screw, but all at the same perpendicular distance from the vertical axis, about which the screw turns; that distance is equal to the radius of the screw, which call r; therefore, $(p + p' + p'' +$ &c.$) r$ is the moment of the whole moving power, which supports $w + w' + w'' +$ &c. or W. But P, acting at the end of the arm GF, produces all this moving power, consequently the moment of P, *i.e.* Pc (if we put c to denote GF), must be equivalent to the former moment.

We have, therefore,

$$Pc = (p + p' + p'' + \text{\&c.}) r =$$

$$r \frac{h + \mu b}{b - \mu h} (w + w' + w'' + \text{\&c.})$$

$$\text{And } \therefore P = \frac{r}{c} \cdot \frac{h + \mu b}{b - \mu h} W.$$

This formula gives the power just necessary to

begin elevating the weight W. Observe, that h is the vertical interval between the threads of the screw, and b the circumference of the cylinder round which the threads run, (see page 307.)

Corollary.—To find the *holding* power, that is, the power just sufficient to prevent W from descending, we may proceed in the same way exactly, only, instead of the equation,

$$AH = AD + \mu(DB + EH),$$

we shall have,

$$AH = AD - \mu(DB + EH).$$

The final result will therefore be,

$$P = \frac{r}{c} \cdot \frac{h - \mu b}{b + \mu h} W.$$

A conclusion, very important practically, follows from this formula; it is this:—As the difference $h - \mu b$ diminishes, P diminishes, and when $h = \mu b$, $P = 0$; hence, supposing μ and b given, it follows that the smaller h is, the smaller the power required to hold W becomes, and when h becomes as small as μb, no holding power is required, no matter how large W may be.

Thus, if $b = 3$ inches, and $\mu = \frac{1}{3}$, no power will be required to hold W, provided h be as small as, (or smaller than,) 1 inch.

CHAPTER VIII.

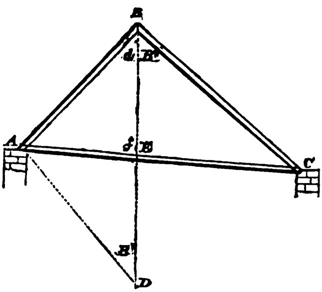
EXAMPLES OF ROOFS.

PROBLEM XXXIX.

To determine the strain produced by the wei a roof upon the tie-beam.

Let AB and BC, fig. 233, be the two bea the roof, resti the walls at A C, and jointe gether at the B; the weig these beams, to with that of slates, laths, &c ing on them tend to depre point B, and out the points

Fig. 233.

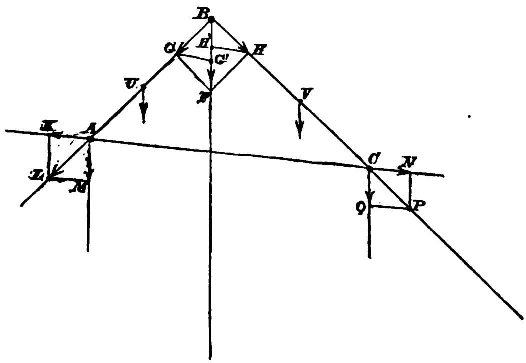


C. To prevent the effect of these outward tl a beam $A\,C$, called the *tie-beam*, is fastened extremities A and C of the two beams of th so as to *tie* them together as it were, and the name. Sometimes the tie-beam is fa at intermediate points of BA and BC, and beams, called the *king-post* and *queen-post* added to give additional strength and stiff

the roof. A great variety of other forms of roof are also very commonly adopted; but we have space only to consider the simplest form here, namely, that shown in fig. 233. The beams AB and BC are almost always of equal length, and the tie-beam horizontal; but this is not assumed to be the case here, for the sake of generality.

Fig. 234 represents the lines of the same roof as that in fig. 233, A, B, and C, being the same points in both figures. Let U and V be the middle points of AB and BC, and let the arrows

Fig. 234.



at U and V represent the weights which rest on AB and BC respectively, namely, the weights of the beams themselves, and the weights of the slates, &c. which are supported by them. These slates, &c. are supposed to be uniformly distributed over the beams, and the beams themselves are also supposed to be uniform in weight and thickness; and hence it is that we place the arrows at the

middle points U and V. We shall denote the weights represented by these arrows by U and V respectively.

Now, U is equivalent to two vertical forces, viz. $\frac{1}{2}U$ acting at B, and $\frac{1}{2}U$ acting at A, and V is equivalent, similarly, to $\frac{1}{2}V$ at B, and $\frac{1}{2}V$ at C; U and V together, therefore, are equivalent to the three vertical forces $\frac{1}{2}(U+V)$ at B, $\frac{1}{2}U$ at A, and $\frac{1}{2}V$ at C. Of these, the latter two, acting immediately on the top of the walls, and vertically downwards, manifestly produce no outward thrust, but are transmitted down the walls to the foundation. These forces, therefore, we may leave out of account, and we have then only to consider the force $\frac{1}{2}(U+V)$ acting at B.

Let the arrow BF represent this force at B, and complete the parallelogram $BGFH$. Then the force BF is equivalent to the two forces BG and BH; and these two forces are transmitted directly along the beams BA and BC respectively, to the points A and C. They are shown in the figure by the arrows AL and CP, which are drawn in the directions of the beams produced, and equal in length to BG and BH respectively. Lastly, the tie-beam AC is produced on both sides to K and N, AM and CQ are drawn vertically, and the parallelograms $AKLM$ and $CNPQ$ are completed.

Now here the force AL is equivalent to the two forces AK and AM, the latter of which acts vertically on the top of the wall at A, and therefore produces no outward thrust. In like manner, the force CP is equivalent to the two forces CN and CQ, of which the latter produces no outward thrust. Thus finally there remain to be considered only

the forces AK and CN, and as these pull directly on the tie-beam in opposite directions, they are the very strains which it is the object of the Problem to determine. *Which was to be done.*

Simplified Construction.—Draw GG' and HH' both parallel to AC; then, in the triangles GBG' and AKL we have AL and BG equal by construction, and the sides of the two triangles respectively parallel to each other; wherefore they are equal triangles, and consequently GG' and AK are equal lines. Hence, by measuring GG' we shall know the strain on the tie-beam at A; and, similarly, by measuring HH' we shall know the opposite strain on the tie-beam at C. It is easy to see that GG' and HH' must always be equal lines, and consequently the strains are not only opposite but equal; which, indeed, might be asserted beforehand as a necessary condition of equilibrium.

Thus, we have the following rule for finding the strains on the tie-beam, viz.:—Draw from B a vertical line BF of the proper length to represent $\frac{1}{2}(U+V)$, *i.e.* half the whole weight of the two beams BA and BC, and the slates, &c. resting on them; then draw from F towards one of the beams AB, the line FG parallel to the other beam CB; lastly, draw GG' from G parallel to the tie-beam, to meet BF at G', measure GG', and the result will be one of the strains required, the other strain being an equal and opposite force.

Corollary.—In fig. 233 draw AD parallel to BC, and BD vertically meeting AC at E; then it is evident, from what has been just said, that we have the following proportion:—

Strain on tie-beam : $\frac{1}{2}(U+V)$ :: AE : BD;

from which proportion the strain may be very easily found, when AE and BD are measured, and $\frac{1}{2}(U+V)$ given.

Mathematical Formula.—Let α, β, and θ, be respectively the angles which the two beams and the tie-beam make with the vertical BD, as is shown in fig. 233. AD being parallel to BC, it is clear that the angle ADB is equal to the angle DBC. Now, by Trigonometry, we have,

$$AB : BD :: \sin.\beta : \sin.(\alpha+\beta)$$
$$AE : AB :: \sin.\alpha : \sin.\theta.$$

Wherefore,

$$AE : BD :: \sin.\alpha \sin.\beta : \sin.\theta \sin.(\alpha+\beta).$$

Hence, by the Corollary we have,

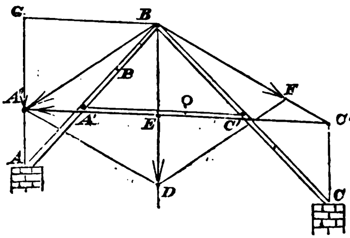
$$\text{strain on tie-beam} = \tfrac{1}{2}(U+V)\frac{\sin.\alpha \sin.\beta}{\sin.\theta \sin.(\alpha+\beta)}.$$

Problem XXXIX.

The tie-beam being fastened at the intermediate points A' and C', fig. 235, it is required to find the strains upon it.

From A and C draw AA'' and CC'' to meet $A'C'$, produced each way, at the points A'' and C'' respectively, and join A'', B, and C'', B. Conceive, as we may do,* that the point A'' is rigidly connected with

Fig. 235.



* See Axioms III. IV. and V.

the beam AB; in short, that AA'' and BA'' are rigid rods, (without weight of course,) connected firmly together at A'', and fastened to AB at A and B, so that ABA'' may be considered as a rigid triangle. In like manner, conceive CBC'' to be a rigid triangle. Draw BD vertically, from A'' draw $A''D$ parallel to BC'', to meet BD at D, and draw DF parallel to BA''; produce AA'' and draw BG parallel to EA'', to meet the produced line at G.

Take BD, as we may do, to represent half the sum of the weights of the beams AB and BC, together with that of the slates, &c. resting on them, (we neglect at present the weight of $A'C'$); then BD is the force, the straining tendency of which it is our object to determine. To do so, we have only to observe that the force BD may be resolved into the two forces BA'' and BF, and these forces may be conceived to act at A'' and C'' respectively, because A'' and C'' are supposed to be points rigidly connected with B. Now the force BA'', acting at A, may be resolved into two forces, represented by the lines EA'' and GA''; the latter of which, GA'', acts vertically on the wall, being transmitted from A'' to A, while the former, EA'', acts directly along the tie-beam. In like manner, we might show that BF'', supposed to act at C'', is equivalent to two forces, one acting vertically downwards on the wall C, and the other directly along the tie-beam. Thus we have the two forces which act directly on the tie-beam, which must be equal and opposite forces, since the tie-beam is in equilibrium. It will be sufficient, therefore, to find one of them, and that we have done.

Hence the following rule for finding the strain on the tie-beam by construction. Having drawn the beams AB, BC, and $A'C'$, in their proper positions, produce $A'C'$ to meet the verticals drawn from A and C at A'' and C''; draw BD vertically cutting $A'C'$ at E, draw also $A''D$ parallel to the line, joining B and C'' to meet BD at D. Then, having measured EA'' and BD, we have the following proportion:—

$$\text{strain on tie-beam} : \tfrac{1}{2}(U+V) :: EA'' : BD,$$

U and V denoting the same as before.

Corollary 1.—*If W denote the weight of $A'C'$, it is required to take it into account.*—We may suppose $\frac{1}{2}W$ to act at A', and $\frac{1}{2}W$ at C', instead of W acting at the middle point of $A'C'$. Also, we may resolve $\frac{1}{2}W$ at A' into two parallel forces, one acting at B, and the other at A. By Cor. 2, p. 168, these forces will be,

$$\tfrac{1}{2}W\frac{AA'}{AB} \text{ at } B, \text{ and } \tfrac{1}{2}W\frac{BA'}{AB} \text{ at } A.$$

Similarly, the force $\frac{1}{2}W$ at C' may be resolved into

$$\tfrac{1}{2}W\frac{CC'}{CB} \text{ at } B, \text{ and } \tfrac{1}{2}W\frac{BC'}{CB} \text{ at } C.$$

The two forces at A and C produce no straining effect, while those at B are to be added to $\frac{1}{2}(U+V)$. Hence we have,

$$\text{strain on tie-beam} :$$

$$\tfrac{1}{2}\left\{U+V+W\left(\frac{AA'}{\mathrm{BB}}+\frac{CC'}{\mathrm{CB}}\right)\right\} :: EA'' : BD.$$

W, AA', CC', AB, and CB being given, we may easily find the strain by this proportion.

Corollary 2.—If W' denote any additional weight acting on one of the beams, AB suppose, at a given intermediate point P, we may find its effect in a similar manner. For it may be resolved into two parallel forces, one acting at B, and the other at A; the former being $W' \frac{AP}{AB}$, which must be included among the forces represented by BD.

Corollary 3.—If W'' denote another additional weight acting at any point Q of $A'B'$, we may find its effect thus. W'' at Q is equivalent to $W'' \frac{QC'}{A'C'}$ at A', together with $W'' \frac{QA'}{A'C'}$ at C'. Again, the force at A' may be resolved into two, one at A and the other at B, the latter being $W'' \frac{QC'}{A'C'} \cdot \frac{AA'}{AB}$. Similarly, the force at C' is equivalent to a force at C, and another at B, the latter being $W'' \frac{QA'}{A'C'} \cdot \frac{CC'}{CB}$. Hence the total force at B arising from W'' is,

$$W'' \left\{ \frac{QC'}{A'C'} \cdot \frac{AA'}{AB} + \frac{QA'}{A'C'} \cdot \frac{CC'}{CB} \right\};$$

and this must be included among the forces represented by BD.

Examples of the preceding Problems.

Ex. 1.—In fig. 233 AC is horizontal, $AB = BC$, $U = V = 1000$ lbs., and $\angle BAC = 45°$; find the strain on the tie-beam.

Ex. 2.—Same case, except that $\angle BAC = 60°$; find the strain.

Ex. 3.—Same case, except that $\angle BAC = 15°$; find the strain.

Ex. 4.—Same case, except that $AC = 50$ feet, and $BE = 1$ foot; find the strain.

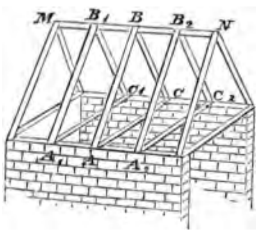
Ex. 5.—In each of these cases what is the total vertical pressure on each wall?

Ex. 6.—Same case as in Ex. 1, except that $AC = 40$ feet, $AB = 30$ feet, and $BC = 20$ feet; find the strain.

Ex. 7.—The angles which AB, BC, and AC make respectively with the vertical, are 30°, 20°, 15°; $U + V = 1000$ lbs.; find strain.

Ex. 8.—Same case as in Ex. 1, except that AB and BC are each 30 feet, and weight each 10 lbs. per foot. Also, the whole roof is composed of 5 equidistant sets of beams, such as ABC, as is shown in fig. 236; there is a beam MN also at the top, weighing 15 lbs. per foot, and 40 feet long, and the slates, laths, &c. weigh 3 lbs. per square foot; find the strain on each tie-beam.

Fig. 236.



Here it is easy to see, that we may suppose $\frac{1}{8}$th of the weight of MN to be supported at M, $\frac{1}{8}$th at N, and $\frac{1}{4}$th at each of the three intermediate

points, B_1, B, B_2. And the weight of the slates, &c. may be distributed similarly. Strictly speaking, however, the distribution of these weights upon the supporting beams, will depend in some degree upon the relative stiffness and strength of the different beams and joinings.

As a good practical rule, it would be well to assume that each set of beams must be strong enough to bear all the weights between it and the two adjacent sets on each side; that is, for example, that ABC (fig. 236) has to support all the slates between $A_1B_1C_1$, and $A_2B_2C_2$, together with the portion B_1B_2 of MN. This will generally overrate the actual strain on each tie-beam, though not the strain that *might be* thrown upon it in consequence of any accidental weakness of the adjacent beams.

The pressure of the wind on a roof is often a serious force, and ought to be taken into account; but it is a point we cannot discuss here. The pressure of the wind is dangerous, not only from its magnitude as a force, but also because it generally acts only on one side of the roof.

Ex. 9.—In fig. 235, A and C are in the same horizontal line, $A'C'$ also is horizontal, and $AB = BC$; if $U+V = 1000$ lbs., $AA' = BA' = A'C'$, find the strain on $A'C'$. (N.B. $W = 0$.)

Ex. 10.—Same case, if $W = 100$ lbs.; find strain.

Ex. 11.—Same case, except that $AA' = 2BA'$, $\angle BA'C' = 45°$, and $W = 100$lbs.; find the strain.

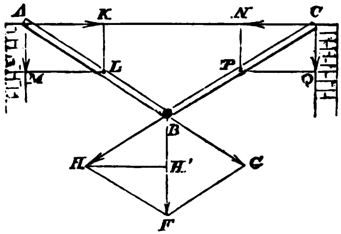
Ex. 12.—Same case as in Ex. 9, except that AA' is so fixed that A' coincides with A, and C' is half way between B and C; also, $\angle ABC = 90°$, and $AB = BC$; find strain.

Problem XL.

To find the inward pull of inverted roof beams on the walls.

Let AB and BC be inverted roof beams, *i.e.* two beams fastened to the walls at A and C, inclining downwards instead of upwards, and jointed together at B. It is required to find the tendency they produce to pull the walls inwards. We suppose that the two beams are equal in length and weight, and that A and C are in the same horizontal line.

Fig. 237.



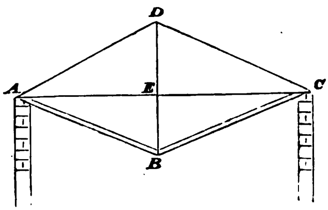
Let U denote the weight of each beam, half of which we may suppose to act at each extremity, *i.e.* $\frac{1}{2}U$ at A, $\frac{1}{2}U$ at C, and $\frac{1}{2}U+\frac{1}{2}U$ (or U) at B; the forces at A and C, as in the former Problems, produce no effect that we are concerned with, and we have therefore only the force U at B to consider. Take BF to represent it, and complete the parallelogram $BHFG$, the two sides BH and BG being in the directions of the beams produced. Take AL and CB equal to BH and BG respectively, and complete the rectangles $AKLM$ and $CNPQ$. Then we may show, exactly as in the previous Problems, that the force BF is equivalent to the forces BH and BG, which again are equivalent to CP and AL, and finally, to the four forces AM, AK, CN, CQ, of which AM and CQ need not be considered.

Thus AK and CN alone are left, and these are the forces which tend to pull the walls inwards. *Which was to be done.*

Simplified Construction.—If we draw HH' horizontally, it is clear that $HH' = AK = CN$. Hence the following construction;—Draw BF vertically of proper length to represent U; draw BH in the direction of CB produced, and FH parallel to AB; lastly, draw HH' horizontally; then HH' being measured, gives the inward pull required.

Corollary.—Draw AD and CD, fig. 238, parallel to BC and BA respectively, and draw BD and AC meeting in E; then we have, as in Cor. page 369,

Fig. 238.

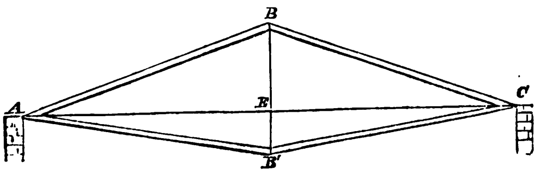


$$\text{Inward pull} : U :: AE : BD.$$

Problem XLI.

To find the horizontal effect upon the walls of the beams AB, CB, AB', and CB', fig. 239, the former being ordinary roof beams, and the latter inverted.

Fig. 239.



Here we suppose that A and C are in the same horizontal line, $AB = CB$, and $AB' = CB'$. Let the weights of AB and CB be each U, and those of AB' and CB' each U'. Also, let T be the *outward thrust* at A (or C), arising from AB and BC; and T' the *inward pull* at A (or C) arising from AB' and CB'. Then, by the former Problems, we have,

$$T : U :: AE : 2BE, \text{ whence } T = \frac{U}{2} \cdot \frac{AE}{BE};$$

and,

$$T' : U' :: AE : 2B'E, \text{ whence } T' = \frac{U'}{2} \cdot \frac{AE}{B'E}.$$

Now $T - T'$ is evidently the horizontal force on the wall at A (or at C); hence the effect required is,

$$\frac{AE}{2}\left(\frac{U}{BE} - \frac{U'}{B'E}\right),$$

which may be easily calculated, when AE, BE, and $B'E$ are determined by measurement, or given.

Corollary.—Hence, if,

$$\frac{U}{BE} = \frac{U'}{B'E}, \text{ or } U : U' :: BE : BE',$$

there is no horizontal effect on the walls.

Problem XLII.

The same being supposed as in the preceding Problem, only that BB' is a bar of given weight V, connecting B and B', it is required to find the crushing force on this bar.

It is clear that there will be a stiffness produced by the bar BB', which will prevent any thrust or pull on the walls, unless BB' yields, and suffers the points B and B' to approach each other. There is, therefore, a tendency to crush BB'. Let the force on BB', which acts at B (downwards) be Y; then that which acts at B' (upwards of course) must be $Y+V$, for it has to support not only the downward force Y, but also the weight of BB', *i.e.* V. We shall suppose, as in the former Problems, that half the weight of each of the four beams, AB, CB, AB', and CB', acts at its extremities. Thus, we shall have at B the total force $U-Y$,* and at B' the force $U'+Y+V$. Hence the outward thrust on the wall arising from the force at B is, by the former Problem,

$$\tfrac{1}{2}(U-Y)\frac{AE}{BE}.$$

And the inward pull at A arising from the force at B' is,

$$\tfrac{1}{2}(U'+Y+V)\frac{AE}{B'E}.$$

Now the outward thrust and the inward pull must be equal, because the effect of the bar BB' is to prevent any pressure on the walls one way or the other. We have, therefore,

$$\tfrac{1}{2}(U-Y)\frac{AE}{BE}=\tfrac{1}{2}(U'+Y+V)\frac{AE}{B'E};$$

$$\text{or, } (U-Y)\,BE'=(U'+Y+V)\,BE.$$

* Y here is *upward*, because it is the resistance which BB' *exerts against the* downward thrust Y on B B'.

From which equation Y may be easily found when BE, BE', U, and U', and V, are given. For example, let these given quantities be respectively 10 feet, 5 feet, 1000 lbs., 200 lbs., 100 lbs.; then the equation becomes,

$$(1000 - Y)\,5 = (300 + Y)\,10.$$

$$\text{And } \therefore\ Y = 133\tfrac{1}{3} \text{ lbs.}$$

Ex. 2.—Same case, only that $BE' = 1$ foot. Here we find,

$$1000 - Y = (300 + Y)\,10.$$

$$\text{And } \therefore\ Y = -181\tfrac{9}{11}.$$

The negative sign here indicates, as usual, that the direction of Y is the reverse of what we originally supposed; that is, that the force exerted on BB' at B is *upwards,* not downwards. This shows that the point B tends to rise, which tendency is prevented by the bar BB'.

CONCLUDING REMARKS.

We have thus finished all that we can at present say regarding the subject of Statics. Many other applications of the principles of this branch of Natural Philosophy are of great importance, some of which are not suitable to this work, and others we must reserve for a future part of it. We now proceed to the subject of *Dynamics.*

PART III.—DYNAMICS.

CHAPTER I.

THE LAWS OF MOTION.

THE object of the science called *Dynamics*, and the meaning of the name, have been explained in Part I. page 73. We shall commence it by a full statement of the *Laws of Motion*, defining, in the first instance, the important term *Velocity*, and explaining the method of compounding and resolving velocities. In what follows, except where the contrary is specified, we shall confine our attention to the motion of a *particle*, or very minute portion of matter; and wherever we speak of a *body* moving, we shall suppose it to be so small, that it may be regarded as a *particle*. (See page 3, Part I.)

The motion of a body consists in its going over or *describing space*—(that is, *linear* or *curvilinear* space, not *superficial* or *solid* space.) The *rate* at which the body moves is an important element of investigation in almost all cases; and this introduces the consideration of *time*. When equal spaces are described in equal times, the motion is

said to be *uniform;* otherwise, it is said to be *variable.* Thus, if a body moves, so that 100 feet are described each minute, the motion is uniform; but if during one minute the space described is 100, during the next minute 110, during the next 120, and so on, the motion is variable.

Velocity.—When a body moves uniformly, the number of feet it goes over in every second is called its *velocity. Velocity* and *rate of motion* are identical terms. Thus, if a body describes 10 feet in every second, its velocity is said to be 10; or it is said to move at the *rate* of 10 feet per second.

Proposition I.

To show that the velocity of a body, which moves uniformly, is equal to the space described in any time divided by that time.

Let s denote the space described in any time t, (i. e., s is the number of feet the body goes over in t seconds;) and let v denote the velocity. Then, v is the space gone over in 1^s; therefore, because the motion is uniform, $2v$ is the space gone over in 2^s, $3v$ the space gone over in 3^s, $4v$ in 4^s; and, generally, tv is the space gone over in t seconds; that is, s. Hence we have

$$s = vt; \text{ and, therefore, } v = \frac{s}{t},$$

which was to be proved.

Hence we may say that the velocity of a body, which moves uniformly, is the space it describes in any time divided by that time; and this is true, no matter how small the time may be; which is

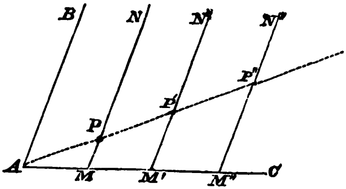
an important consideration to be borne in mind, as will appear when we come to speak of variable motion.

Composition and Resolution of Velocities.—That velocities may be resolved and compounded in the same manner, and by the very same rules as forces, we shall now prove.

Proposition II.

To find the motion of a body which has two different velocities communicated to it at the same time.

Fig. 240.



Let AB and BC, Fig. 240, be two fixed lines, making any angle with each other, MN a moveable rod, which always continues parallel to AB, while the extremity M moves along AC, occupying successively the positions M, M', M'', &c. Let P be a body capable of moving along this rod; and suppose that, while the extremity M is moving along AC, P moves along the rod, starting from its extremity, occupying successively the positions P, P', P'', &c.

This being the case, it is clear that the body P has two different velocities communicated to it at the same time; for all the points of the rod, and therefore P, have the same motion as M; also P has, in addition to this, a motion along the rod. P, then, has, at the same time, two distinct velocities, namely, its own velocity along the rod,

and a velocity parallel and equal to that of M along AC; that is, P has a velocity parallel to AB, and a velocity parallel to AC, communicated to it at the same time.

Now, the actual motion of P is evidently along the oblique line $APP'P''$, which may be either a straight line or a curve, according to circumstances. If the motions of M along AC, and of P along the rod, be uniform, MP will be always proportional to AM; that is, we shall have

$$MP : AM :: M'P' : AM' :: M''P'' : AM'' \text{ \&c.}$$

For instance, if $MP = 2AM$, then will $M'P' = 2AM'$, $M''P'' = 2AM''$, &c.; or, if $MP = \frac{1}{3}AM$, then will $M'P' = \frac{1}{3}AM'$, $M''P'' = \frac{1}{3}AM''$, &c.; and so for any other number.

In this case, $APP'P''$ &c. will be a straight line; for it is the peculiar property of a straight line, as we know by Euclid, Book VI., that the parallel lines MP, $M'P'$, $M''P''$, &c. increase proportionally to the distances AM, AM', AM'', &c. It is manifest, also, that the actual motion of P along the distances $APP'P''$ is uniform, since the distances AP, AP', AP'', are proportional to AM, AM', AM'', &c., or the distances MP, MP', MP'', &c. respectively.

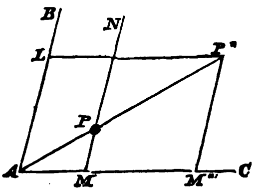
If we suppose that M moves from A to M'' in one second of time, AM'' will be, by definition, the velocity of M; and since, in the same time, P moves along the rod as far as P'', $M''P''$ will be the velocity of P along the rod. Furthermore, AP'' being the space actually described by P in a second, the actual velocity of P will be AP''.

Hence we have the following rule for finding the actual velocity of P arising from the two

velocities parallel to AC and AB. Measure along AC a space AM'', equal to the velocity which P has parallel to AC; draw $M''N''$ parallel to AB, and measure along it a space $M''P''$, equal to the velocity which P has parallel to AB; then join A and P'', and the line AP'' will represent the actual velocity of P in magnitude and direction; that is, P really moves in the direction of the line AP'', and describes a space equal to AP'' every second.

Corollary 1.—We may state this rule in the following manner also:—Measure a space AM'' along AC (fig. 241) equal to the velocity of P parallel to AC; also measure a space AL along AB equal to the velocity of P parallel to AB; draw $M''P''$ parallel to AB, and LP'' parallel to AC, so forming a parallelogram, $ALP''M''$; then, drawing the diagonal AP'', AP'' will be the actual velocity of P in magnitude and direction. This is manifestly equivalent to the former rule, inasmuch as $M''P''$ is here drawn parallel to AB, and equal to the velocity of P parallel to AB.

Fig. 241.



Corollary 2.—It appears, then, that the actual velocity of P is found in exactly the same way as the resultant of two forces, in Statics, by the Parallelogram of Forces. We may call AP'' the *resultant* of the two velocities AM'' and AL; we may call AM'' and AL the *components* of the velocity AP''; and we may speak of the *composition* of the two velocities AM'' and AL into a

single resultant velocity AP'', or of the *resolution* of the velocity AP'' into the two component velocities AP'' and AL.

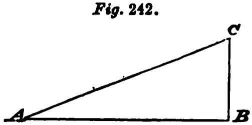
Thus it follows that velocities may be resolved and compounded in exactly the same way as forces in Statics, and by the very same rules and formulæ. It is, therefore, unnecessary to pursue this subject farther, inasmuch as the rules for the composition and resolution of forces already given in *Statics*, may be transferred to the composition and resolution of velocities in *Dynamics*, without the least alteration, except the substitution of the word *velocity* for the word *force*.

Examples.

Ex. 1.—A particle has a horizontal velocity of 10 feet per second, and at the same time a vertical velocity of 10 feet per second; find its actual motion.

Draw AB, fig. 242, horizontally equal to 10, and BC vertically equal to 5; join A and C; and AC will be the actual velocity in magnitude and direction. AC, and the inclination CAB, may be calculated or measured.

Fig. 242.



Ex. 2.—What is the actual velocity when the horizontal velocity is 20, and the vertical 50?

Ex. 3.—A body moves along a plane inclined at 30° to the horizon with a velocity 10; what are its horizontal and vertical velocities?

Draw AC equal to 10, making $\angle CAB = 30$, and draw CB perpendicular to AB; then AB

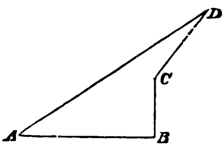
and BC will be the required horizontal and vertical velocities.

Ex. 4.—The vertical velocity of a body is 5 times its horizontal velocity; at what angle to the horizon does it actually move?

Ex. 5.—A body has three velocities communicated to it at the same time; one a horizontal velocity 30, the second an upward vertical velocity 10, the third a velocity 20 inclined upwards at 45° to the horizon; what is the actual velocity?

This must be found in the same manner as the resultant of three forces, by the Polygon of Forces in Statics (see p. 145); that is, draw AB, fig. 243, horizontally equal to 30, BC vertically equal to 10, and CD at 45° to the horizon equal to 20; then, joining A and D, AD will be the actual velocity in magnitude and direction.

Fig. 243.



OF THE THREE LAWS OF MOTION.

The Laws of Motion are certain statements respecting the effect of force in producing motion, upon which all the subsequent reasoning in Dynamics depends. We shall now enumerate and explain these laws, and the manner in which they have been proved.

FIRST LAW OF MOTION.

When a body is not acted upon by any force, if at rest, it remains at rest; and if in motion, it continues to move uniformly in the same direction.

We have little difficulty in receiving the first part of the statement made in this law, namely, that if a body at rest is not acted upon by any force, it remains at rest. Not so, however, as regards the second part of the law; for our common notion is, that a body cannot *continue* to move without the help of some force to keep it in motion, and that it must immediately stop if the force ceases to act upon it. Now this is quite an erroneous idea, and the first law of motion, without the last sentence, ("*and, if in motion, continues to move uniformly in the same direction,*") would be only half the truth. The fact is, that a body once put in motion will persevere in that motion without the help of any force, and it is as natural for a body to be in motion as to be at rest. If a body is at rest, its natural condition is rest, and it will not move without the action of some force to disturb it. If a body be in motion, its natural condition is motion, and it will not come to rest without the action of some force to stop it.

But it may be said, that all bodies appear to have a tendency to come to rest. Thus, if we put a train in motion along a level railroad, it will soon stop, except the force of the engine keep it in motion. This tendency to come to rest, is not, however, anything inherent in the body, but is the effect of certain retarding forces, which we overlook. In the case of the train there are two retarding forces, one arising from the resistance of the air, the other from the friction of the axles. These together produce a serious amount of resistance, and a consequent tendency in the train to stop. The force of the steam is required to overcome these resistances; when it is

just sufficient to do this the train moves uniformly, when it is more than sufficient the motion becomes continually quicker, and when it is not sufficient the motion becomes continually slower, and at length ceases.

When the causes of resistance are diminished, it is found that less force is required to keep up the motion; thus, if the friction of the axles be diminished, by making them smaller and harder, and rendering them smoother by the application of an unguent or grease, the force of steam required to keep up the motion of the train is diminished in proportion; so much so, that we are justified in concluding, that, if we could destroy altogether the friction, and the resistance of the air, the train, once put in motion, would continue to move with an undiminished velocity, without any steam force to keep up the motion.

We conclude, then, that bodies once put in motion have no inherent tendency to come to rest, such tendency being the effect, in all cases, of certain resisting forces, of which we do not take notice at first sight. The first law of motion, as above stated, is therefore, to a certain extent, proved by these considerations. It is, however, to other considerations connected with Physical Astronomy, which we shall presently mention, that we must look for a complete proof of this law.

Inertia.—The word *inertia* is often used in Mechanics to denote the property of matter, asserted in the first law of motion. Matter is said to be *inert*, which word, in its proper signification, means, devoid of any power of moving *without the* action of some external force. It

must be taken, however, in a larger sense than this, for matter has also the property of continuing in motion when once put in motion. By inertia, therefore, we mean, not simply nor properly, a tendency to remain at rest, but a tendency to remain at rest or in motion, as the case may be. Inertia, therefore, means not any *sluggishness* or tendency to stop, but a power of resisting change, whether it be from rest to motion, from motion to rest, or from one motion to another. If a body be at rest, it has a tendency to remain at rest, and a power to resist any moving force; if it be in motion, in a certain direction, and with a certain velocity, it has a tendency to persevere in that motion, as regards both the direction and velocity, and a power to resist any force tending to change that direction or velocity.

Vis inertiæ.—This power of resisting change, from rest to motion, from motion to rest, or from one motion to another, is usually called the *vis inertiæ*, or the *power of inertia*. It is found by experiment, that the vis inertiæ of any body is proportional to its weight; but of this we shall say more presently.

From what has been said, it is clear that "*inertia*" is not exactly the proper word for denoting the peculiar tendency of matter described in the first law of motion. *Inertia* means *laziness*, and seems to imply, therefore, the necessity of some force to keep up motion. But, if I may so express myself, the first law of motion does not charge matter with laziness, but with *obstinacy;* it asserts that matter persists in motion or rest indifferently; that it has no choice, so to speak, but simply a tendency to continue in that state

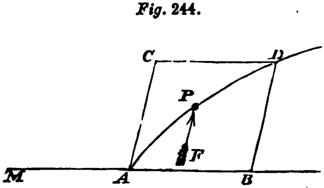
in which it is. Hence the word *pertinacia*, or *pertinacity*, would be more suitable than *inertia*. The term *vis inertiæ* is sanctioned by use, but it appears to be a highly objectionable expression, and I could mention several cases in which it has led to error.

SECOND LAW OF MOTION.

When a body, moving in a certain direction, is acted on by a force oblique to that direction, the deflection produced in any time is equal to the space which the body would have described in that time, if it had been originally at rest.

Let MAB, fig. 244, be the direction in which the body P was originally moving; suppose the force F to begin to act upon P at A, always parallel to the line AC, and to *deflect* or divert P out of its rectilineal course, so as to make it describe the path APD. Draw DB parallel to AC, and DC parallel to AB. Let t denote the time the body takes to move from A to D.

Fig. 244.



As we have shown above, we may consider the motion APD to be the resultant of the two motions AB and AC, of which AB is in the direction of the original velocity, and AC in the direction of the force F. BD is said to be the *defection* produced in the time t by the force, and AC is equal to BD. Now the second law of

motion amounts to this, that AC, or BD, is equal to the space the body would have described in the time t by the action of the force, if there had been no previous velocity in the direction MAB.

We may state the law more distinctly as follows:—

Question.—To find where the body P will be at the end of any time t?

Answer.—Make AB equal to the space that P would describe in the time t, if F did not act, and AC equal to the space that F would make P describe, if P had no previous velocity; then, if we draw BD and CD parallel respectively to AC and AB, to meet at D, D will be the place where P is at the end of the time t.

In other words, the *actual motion* of P is *the resultant of two motions,* one, that *due to the previous velocity simply,* the other, that *due to the action of F simply.*

Experimental proof of the second law of motion.

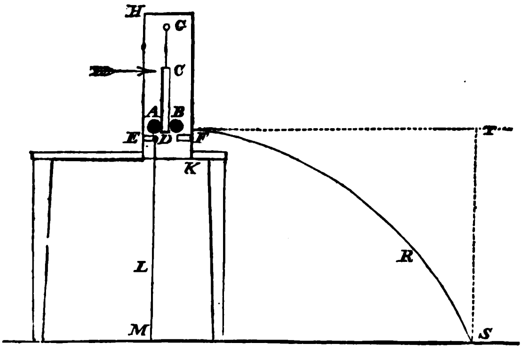
The following simple experiment forms a good proof of the second law of motion:—

Fix a piece of board, HK, fig. 245, vertically to a table, pillar, wall, or otherwise; fix two small projecting pieces of wood, E and F, to HK; hang another piece of wood CD, by a weak spring or wire GC, from a nail or projection G, fixed in HK; place two balls A and B resting partly on the projections E and F, and partly against the piece CD, as is shown in the figure.

The balls being thus placed, strike CD smartly with anything hard, in the direction represented by the arrow. The effect of the blow will be to throw the ball B forwards, and make it describe

a curve *BRS*, striking the floor at *S*; but the ball *A* will be let drop vertically downwards in the direction *ALM*, striking the floor at *M*.

Fig. 245.



Now, if the floor be horizontal, and the apparatus properly adjusted so as to make *CD* strike *B* horizontally, it will be perceived by the ear, that the two balls strike the ground exactly at the same instant, and this will be true at whatever height above the floor the apparatus may be fixed.

If the ball *B* were not drawn downwards by the force of gravity, it would describe a horizontal line *BT*, because it is struck in a horizontal direction, as we have stated; but the force of gravity acting downwards, gradually deflects it out of the horizontal direction, and makes it describe the curve *BRS*. If we draw *ST* vertically, *ST* is the defection produced by the force of gravity.

Now *ST* is manifestly equal to *AM*, and *AM* is the space which the force of gravity makes a body having no previous velocity describe in the

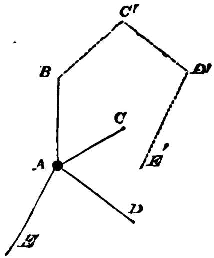
same time that the ball B moves from B to S. It appears, therefore, that the defection TS, produced by the force of gravity in a body having a previous horizontal velocity, is equal to the space BM, which a body, without a previous velocity, describes by the action of the same force in the same time. This is the second law of motion.

As we have observed before, experiments of this kind can only be adduced as rough proofs, or rather, instances of the laws of motion; the general and accurate proof we shall presently state. In the present case, we have not taken any account of the force arising from the resistance of the air, because the experiment is not capable of very great nicety.

Extended statement of the second law of motion.

We may extend the statement of the second law of motion, as follows;—Suppose A to be a body moving with a certain given velocity, and acted on by several given forces; it is required to find where A will be at the end of any time t.

Fig. 246.



To determine this, let AB, fig. 246, be the space the body would describe in the time t, in consequence of the given velocity, supposing no force to act on it; also, let AC, AD, AE, be the respective spaces which each force by itself, and without a previous velocity, would make the body describe in the time t; then the actual

motion of the body will be the resultant of the combined motions AB, AC, AD, and AE; and therefore, if we draw BC' parallel and equal to AC, $C'D'$ parallel and equal to AD, $D'E'$ parallel and equal to AE, E' will be the actual position of the body at the end of the time t.

Observe, we find the point E' as if we were finding the resultant of a set of forces, by the rule given in Statics. See Dynamics, page 386.

This general and extended enunciation of the second law of motion, is very important and useful. It may be summed up in the following words:—

A set of motions which take place simultaneously, may be supposed to take place successively.

For AB, AC, AD, AE, are a set of motions, which take place *all at the same time*, or *simultaneously*; AB, BC', $C'D'$, $D'E'$, are the same motions supposed to take place *one after the other*, or *successively*; and E' is the actual place of the particle at the end of the combined motions, whether they be simultaneous or successive.

THIRD LAW OF MOTION.

The velocity produced by a force acting on a given body, during a given time, is proportional to the force.

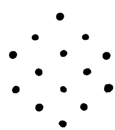
Let P be the force, W the body on which it acts for a certain time, and v the velocity it produces in that time; then the third law of motion is, that v is proportional to P. Thus, suppose that a force of 6 lbs. produces a velocity of 12 in a second; then 12 lbs. will produce a velocity 24, 18 lbs. a velocity 36, 3 lbs. a velocity 6, 1 lb. a

velocity 2, &c. &c. And in general, if v and v' are the velocities produced in the same time by the forces P and P', acting on the same body, we have,

$$v : v' :: P : P'.$$

Velocity produced by Gravity.—It is found by experiment, that the force of gravity produces in a given time, the same velocity in every body allowed to fall freely, whether small or large, light or heavy. This is not true when bodies are allowed to fall in the air, because the air has the power of resisting motion to a certain extent, and this resistance, for obvious reasons, tells more upon a light body than upon a heavy. But if bodies however different, as for example, a feather and a piece of gold, in the well-known experiment of the air pump, be allowed to fall in vacuum, it is found that they move downwards with exactly the same degree of rapidity.

Fig. 247.



That this ought to be the case may be easily shown as follows:—Let the dots in fig. 247 represent a set of particles of equal size and weight, not connected with each other. If they be allowed to fall, they will move downwards all at the same rate, and therefore keep always at the same distance from each other. This being the case, it is no matter whether we suppose them to be rigidly connected with each other or not; we may therefore assume that they are rigidly connected with each other, and so constitute a rigid body. It appears, then, that a rigid body falls down at the same rate its particles would fall, if they were disconnected and separate;

and this is manifestly true, whether the particles be close together or far apart, *i.e.* whether the body be condensed, and therefore heavy, or expanded, and therefore light.

When a body is allowed to fall in vacuum, it is found that at the end of 1 second it has a velocity of about 32.2, at the end of 2 seconds it has a velocity of 64.4, at the end of 3 seconds a velocity of 128.8, and so on. That is, the body is moving at the rate of 32.2 feet per second at the end of the 1st second, with a velocity of 64.4 feet per second at the end of the 2d second, and so on. In half a second the velocity acquired is 16.1, in a quarter of a second 8.05, and so on.

At the same time it is found, that the space the body falls down in 1 second is 16.1 feet, in 2 seconds the space is 4 times as much, in 3 seconds 9 times as much, in 4 seconds 16 times as much, and so on; the space being always proportional to the square of the time.

To express this algebraically, suppose the body is allowed to fall from A, fig. 248, that it reaches B in t seconds, and that the space AB is s feet; then,

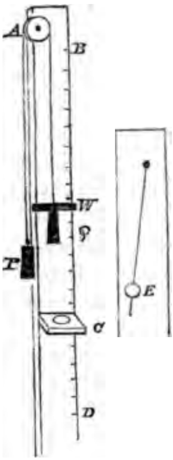
Fig. 248.

$$v = 32.2 \times t, \quad s = 16.1 \times t^2.$$

The truth of this formula may be proved by experiment, though not directly, for the motion of a falling body is so quick, that it is difficult to observe it with accuracy. An apparatus, called Atwood's Machine, is often employed to show the truth of these formulæ. It consists of a wheel, or fixed pulley A, fig. 249, made so as to turn very smoothly; over

this a string passes, to ends of which are attached two equal weights P and Q. A third weight W, of an oblong form, with a hole in the middle through which the string passes freely, is placed on the top of W. BD is a vertical scale, divided into feet and inches. At the side there is a stage C, capable of being fixed at any elevation along the scale. In this stage is a hole large enough to let the weight Q pass, but sufficiently small to stop W. At the side is a pendulum E beating seconds.

Fig. 249.



This apparatus is used in the following manner, to determine the motion produced by a force putting a certain quantity of matter in motion. Q, with W resting on the top of it, is drawn up to the top of the scale B, and allowed to fall. The number of seconds occupied in moving from B to C, is observed by means of the pendulum; also, how far Q moves in one second after passing through the hole at C is observed.

Now it is evident that $P + Q + W$ is the quantity of matter moved in this case, supposing we take no account of the motion of the wheel A; also, since P and Q, being equal, balance each other, W is the force which puts this matter in motion. Again, if CD be the space Q moves down in 1 second, after passing through the hole, CD is the velocity acquired in moving from B to

C; for, while *Q* is moving from *C* to *D*, *W* is stopped by the stage, and there is no moving force; consequently, the motion from *C* to *D* is uniform (first law of motion), and therefore *CD*, being the space described in one second, is the velocity of *Q* at the instant when *W* is stopped by the hole. A certain allowance, however, must be made for the motion of the matter composing the wheel *A*. By making *W* sufficiently small, we may make the motion as slow as we please, and so observe it more accurately; this is, in fact, the peculiar advantage gained by the apparatus. But as far as accuracy is concerned, this machine is a mere philosophical toy, with the disadvantage of requiring a considerable knowledge of Dynamics, to understand the motion properly; for the motion of *A* must be allowed for in a manner which a beginner cannot understand, especially if there be the addition of what are called *friction wheels*, to lessen the retarding power of the friction of the pivot, about which *A* turns.

The results that would be obtained by means of this machine, if the wheel *A* had no weight, and its axle were perfectly smooth, are as follows:—

$$s = BC = 16.1 \frac{W}{P+Q+W} t^2.$$

t being the time of moving from *B* to *C*,

$$\text{and } v = CD = 32.2 \frac{W}{P+Q+W} t.$$

Whence it appears, that the motion produced by a moving force *W* is directly proportional to

W, and inversely proportional to the total weight moved, namely, $P + Q + W$.

From these results we may conclude, that if the moving force were equal to the total weight moved, as is the case with a body falling freely in vacuum, we should have,

$$s = 16.1\, t^2, \quad v = 32.2\, t.$$

But the accurate method of proving these formulæ for falling bodies, is by experiments with pendulums, as we shall show hereafter, when we come to speak of the pendulum. The number 32.2 is that usually given, but it is not quite exact, being slightly different at different places on the earth's surface, and at different elevations above the level of the ocean.

Proposition III.

To determine, by the third law of motion, what velocity is produced in a second by a given force, expressed in pounds, acting on a given weight, expressed in pounds likewise.

Let P be the number of pounds in the force, W the weight, in pounds, of the body on which it acts, and f be the velocity produced in 1 second. Then, if W were allowed to fall freely by the action of its own weight, the velocity produced in a second would be 32.2, as appears by experiment; that is, a force W acting upon a body whose weight is W, produces a velocity 32.2 in 1 second. Hence, f and 32.2 being the velocities produced in the same time by the forces P and W respectively, acting on the same body, we have, by the third law of motion,

$$f : 32.2 :: P : W.$$

Whence $f = \frac{P}{W}\, 32.2$, which is the required velocity.

It appears from this that the velocity which a force produces in a second, is found by multiplying the force by 32.2, and dividing it by the weight of the body it acts upon.

Corollary.—By the proportion just obtained, we find also, that,

$$P = \frac{W}{32.2} f.$$

That is, the force necessary to produce a certain velocity in a unit of time, is found by multiplying the weight of the body on which it acts by the velocity, and dividing by 32.2.

Meaning of the letter g in Dynamics.—We have stated above that there is some degree of difference in the force of gravity at different places, and at different elevations; the number 32.2 employed in the above formula is not therefore quite correct in all cases. On this account it will be better to use some letter instead of 32.2, and the letter always employed is g. We define g therefore to be the velocity which is produced by the attraction of gravity at a given place on the earth's surface in one second. The exact value of g in the latitude of Greenwich is 32.1908, which is nearly the same thing as 32.2.

Hence the force, which, acting on a body weighing W lbs., produces a velocity f in a second, is,

$$\frac{W}{g} f;$$

and the velocity which is produced in a second

by a force P, acting on a body weighing W lbs. is,

$$\frac{P}{W}g.$$

These formulæ are of great importance in Dynamics.

EXAMPLES OF THE APPLICATION OF PROPOSITION III.

Ex. 1.—A body weighing 100 lbs. (W) is drawn along a perfectly smooth horizontal plane by a horizontal force of 10 lbs. (P); find the velocity generated per second.

Here $P = 10$, $W = 100$.

$$\therefore f = \frac{P}{W}g = \frac{10}{100} \times 32.2 = 3.22.$$

Ex. 2.—In the same case find f, supposing the horizontal plane to be rough, and the coefficient of friction $\frac{1}{20}$ (μ).

$$\text{Here } P = 10 - \frac{1}{20}W = 5.$$

$$\therefore f = \frac{5}{100} \times 32.2 = 1.61.$$

Ex. 3.—In the same case as Example 1, only that P is unknown; find what P must be to make the body move twice as fast as a falling body moves.

Ex. 4.—In the same case as Example 2, only that μ is unknown; find μ, supposing that a falling body moves 100 times faster than W.

Ex. 5.—A body (W) is drawn up a smooth inclined plane, whose gradient* is $\frac{1}{100}$, by a force F acting along the plane; if F is the tenth part of W, find how much slower the body moves than a falling body.

Here resolve W into two forces, one along the plane, and the other perpendicular to it; the former must be subtracted from P, and the latter produces no effect as regards the motion. Thus we find,

$$P = F - \frac{1}{100} W = \frac{9}{100} W.$$

$$\therefore f = \frac{P}{W} g = \frac{9}{100} g.$$

Hence the body moves slower than a falling body in the proportion of 9 to 100.

Ex. 6.—In the same case as Example 5, supposing the body to move slower than a falling body in the proportion of 3 to 5; find F.

Ex. 7.—The gradient in Example 5 being $\frac{1}{2}$, and $P = 10\,W$; find how much faster the body moves than a falling body.

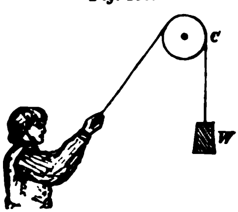
Ex. 8.—The length, height, and base of a smooth inclined plane are respectively 5, 4, and 3, and a body is allowed to run down it freely; find how much slower it moves than a falling body.

Ex. 9.—Find the same supposing the plane to be rough, and the coefficient of friction $\frac{1}{8}$.

* The gradient of a slope is the rate of vertical ascent, and may be expressed by saying, that a body rises vertically a certain fraction of a foot for every foot it travels up the plane. In the Example the body rises the hundredth part of a foot for every foot it goes up the plane.

Ex. 10.—A man holds a weight W by a string WC, fig. 250; if he relaxes the string so as to let the body move downwards slower than a falling body in the proportion of 1 to 3, find the tension on the string CW.

Fig. 250.



Here let T be the tension on the string; then $W-T$ is the force acting downwards on the body.

$$\therefore \; W-T=P=W\frac{f}{g}=\tfrac{1}{3}W \;(\because f=\tfrac{1}{3}g).$$

$$\therefore \; 3W-3T=W, \text{ and } \therefore \; T=\tfrac{2}{3}W.$$

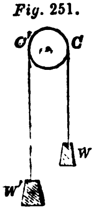
Ex. 11.—Suppose that the man draws the string so as to make the body move up as fast as a body falls down naturally; find T.

$$\text{Here } T-W=P=W\frac{f}{g}=W \;(\because f=g).$$

$$\therefore \; T=2W.$$

Ex. 12.—The weight W' draws up the weight W, as in fig. 251, the pulley CC' being supposed to be perfectly smooth, so that whatever tension is exerted on one portion $C'W'$ of the string is transmitted unaltered to the other portion CW, and *vice versâ*. If $W'=100$lbs. $W=50$lbs. it is required to find the tension on the string.

Fig. 251.



Here let T be the tension on WC, and therefore on $W'C'$, and suppose that W' descends with

x times the speed of a falling body. It is clear that W must ascend just as fast as W' descends; also, the force that urges W upward is $T-W$, and that which urges W' downward is $W'-T$. Hence we have,

$$T-W=W\frac{f}{g}=Wx\ (\because f=xg),$$

$$\text{And } W'-T=W'\frac{f}{g}=W'x;$$

$$\therefore \text{ by division, } \frac{T-W}{W'-T}=\frac{W}{W'}, \text{ or } \frac{T-50}{100-T}=\frac{1}{2}.$$

$$\therefore\ 2T-100=100-T.$$

$$\therefore\ T=\frac{200}{3}=66.6.$$

Ex. 13.—Find x in Example 12, W' and W being any weights.

We have, $\qquad T-W=Wx,$

And, $\qquad W'-T=W'x.$

$$\therefore \text{ by addition, } W'-W=(W'+W)x.$$

$$\therefore\ x=\frac{W'-W}{W'+W}.$$

Thus, when $W'=100$, and $W=50$, $x=\frac{50}{150}=\frac{1}{3}$. When it appears that W' descends with one-third of the speed of a falling body.

Ex. 14.—Find T in Example 13.

$$\text{By division, } \frac{T-W}{W'-T}=\frac{W}{W'};$$

$$\therefore\ W'T - WW' = WW' - W'T;$$

$$\therefore\ T = \frac{2WW'}{W + W'}.$$

Ex. 15.—The tension on the string is equal to $\frac{3}{2}W$; find in what proportion W' exceeds W.

Ex. 16.—Show that the tension can never be greater than double the lighter weight.

Ex. 17.—Find in what proportion the total pressure on the pulley CC' is less than the sum of the weights, when $W' = 5W$.

Ex. 18.—When W' is extremely large compared with W, show that T is very nearly twice W.

GENERAL REMARKS ON THE LAWS OF MOTION.

Action and Reaction.—The laws of motion above enunciated are not exactly those given by Newton, to whom chiefly we owe all our knowledge on this subject. It is customary with many writers to divide the laws of motion as above, and perhaps it is on the whole a better division. Newton's *second* law virtually includes the second and third laws as we have stated them, and his *third* law is this:—*Action and Reaction are equal and opposite.* That is, if one body A exerts a force upon another body B, by contact, by tension or thrust, by attraction or repulsion, or otherwise; then B exerts the very same force on A in the opposite direction; in other words, the *return pressure or reaction* (see Part I. page 32) of B on A is equal and opposite to the *original pressure*, or action of A on B. This is a most important mechanical law, but it belongs as much to Statics as Dynamics,

and is not properly therefore to be regarded as a special law of motion. As an example of this law, we may quote the case of the mutual attractions of the sun and planets; thus the actual force of attraction which the sun exercises on the earth, is equal to that which the earth exercises on the sun. This may appear strange, considering how much smaller the earth is than the sun, but it is nevertheless the fact.

Proof of the Laws of Motion.—The proof of the laws of motion is of that kind which is called *inductive*. By the term *induction* is meant the verification of conjectured laws, by numerous particular instances of their truth; or to speak more fully, I conjecture from certain appearances that a certain law is true, and I set about establishing that truth, by investigating the consequences that must follow from the law in particular cases, and observing whether these consequences are really matters of fact. If so, and if I have tested the law by a sufficient number and variety of particular cases to prevent the risk of mistake, I feel satisfied that it is true. This is reasoning by induction. This, however, is not the sense in which the word induction was used in ancient times. We owe the term, as it is generally used at present, to the great philosopher Lord Bacon.

Inductive reasoning is the grand basis of all true science, and were it not for the instinctive confidence which we feel in it, we should know nothing worth knowing. It is by induction that we come to the conclusion that the earth is a sphere revolving about the sun, that the planets revolve about the sun, that there is a force called the attraction of gravitation, &c. None of these

things can be proved by demonstration like a proposition of Euclid; but we believe them all most implicitly. Now we cannot avoid remarking here, that the evidence we have of the existence of God, of his righteous government, and of the Revelation he has made in the Holy Scriptures, is precisely of the same inductive nature; and we have the same grounds for receiving the great truths of Christianity, that we have for admitting the laws of motion. Those who contrast the truths of Religion and Science, as if the latter were based upon higher evidence than the former, only betray their utter ignorance of the nature of human intellect, and the true foundation of science.

The proof of the laws of motion consists chiefly in the exact coincidence between the deductions of Physical Astronomy, and the observed phenomena of the heavens. Physical Astronomy is that science which, assuming the laws of motion as true, determines from them the motions of the heavenly bodies; and this it has done, as regards the solar system, with a most wonderful degree of minuteness. Having thus determined the motions of the planets and their satellites, it predicts their relative positions at future times, their eclipses, transits, occultations, &c. Now, the practical astronomer with his telescope finds these predictions marvellously exact; and hence we infer the truth of the laws of motion, for it is upon them that Physical Astronomy bases all its calculations.

But, the reader may ask, are the predictions we speak of never wrong? are they always exactly fulfilled? Our answer is, they are sometimes wrong. Why, then, do we so implicitly receive

them as evidence to prove the laws of motion? Because, for one case in which they are wrong, they are right in a thousand cases, and the true philosopher knows well that the errors may, and are likely to arise from imperfections in his own knowledge or means of investigation. To quote an instance:—Not many years since the predictions of Physical Astronomy, in regard to the motion of the planet Uranus, were found to be failures; the planet was not apparently obedient to the laws of motion. Did astronomers then reject the laws of motion? Far from it; but they came to the conclusion that there was some unknown force interfering with Uranus. And, with the utmost confidence in the laws of motion, they employed them to discover what that unknown force was, and where it resided. It is unnecessary to dwell on the result of this confidence in these fundamental laws; a new planet was discovered by the mind's eye alone, and the telescope confirmed the great discovery.

CHAPTER II.

CONSEQUENCES AND PRINCIPLES IMMEDIATELY DEDUCIBLE FROM THE LAWS OF MOTION.

PROPOSITION IV.

To find the velocity which a given force, acting on a body of given weight, produces in a given time.

Let P be the given force, W the given weight of the body, and t the given time expressed in seconds; let f be the velocity produced in one second, that is,

$$f = \frac{P}{W} g.$$

This velocity is produced in a body previously at rest; but, by the second law of motion, the velocity produced by the force, if the body had a previous motion, would be just the same; so that, if the body started with a velocity v, the force would produce in a second an additional velocity f, and the total velocity at the end of the second would be $v + f$.

Hence it is clear that the velocities at the end of the first, second, third, and fourth seconds will be respectively f, $f + f$, $f + f + f$, $f + f + f + f$;

on. If, therefore, v be the velocity at the t seconds, it is clear that

$$v = ft, \text{ or } v = \frac{P}{W} g t.$$

ppears, then, that the velocity produced in ne t is found by multiplying the velocity ɔed in one second by t.

ɔlerating Force.—It is usual, in Mechanical ses, to designate the velocity produced in a . by the term "*Accelerating Force;*" but we er this to be a very bad term for the purpose, generally misleads the student as to the ature of the quantity denoted. We shall use this term, but speak of f simply as the *ity generated per second,*" or "*rate of accele-*"

ss.—The term *mass* is used to denote the ty of matter in a body, as indicated by its t. It is generally measured with reference ɔculiar unit in Mechanical Treatises, for the e of making the above expression for f as as possible. The unit of mass is considered a body weighing 32.2 lbs., or more correctly, g, as we have stated, being (at Greenwich) 08. The quantity of matter in a body ng g lbs. is therefore assumed to be 1, in weighing $2g$ lbs. it is assumed to be 2, , 3, and so on; and in general, the mass ody whose weight is Mg lbs. is assumed to If, then, W be the weight of a body, and mass, we have,

$$W = Mg.$$

Now, in the formula, $f = \frac{P}{W} g$, put this value of W, and we obtain,

$$f = \frac{P}{M}, \text{ and } \therefore P = fM.$$

That is, the velocity per second is found by dividing the force by the mass, and the force is found by multiplying the velocity per second by the mass.

Why the unit of mass is assumed to be g lbs.—There is a twofold reason for choosing this particular weight as the unit of mass. First, because it simplifies the expression $\frac{W}{g} v$, and certain other formulæ, which frequently occur in Dynamics, inasmuch as it represents the fraction $\frac{W}{g}$ by the single letter M. And secondly, because it gives an estimation of the mass of a body which answers equally well for all places in the universe, as far at least as the laws of motion extend. On this second reason it may be well to speak more explicitly. The weight of a body is not an invariable quantity, inasmuch as it depends upon the force of attraction of the planet in the immediate vicinity of which the body may happen to be, and its distance from the centre and axis of rotation of that planet. Thus a body is heavier at the pole of this earth than at the equator, heavier also at the level of the sea than at the top of a mountain. An ordinary man transported to the sun, would weigh some two tons, but if placed on one of the small planets, his weight would not exceed one stone. Now our notion of the matter

body is, that it is the same in amount ever the body may be. Hence it follows that weight may vary though the quantity of ,er remains the same. We cannot therefore nate quantity of matter by weight unless all ies be supposed to be weighed at the same ;e. The simplest way to get over this difficulty, o to vary the unit of mass, that it shall always rease or diminish in the same proportion that ; weight of a body does in consequence of a ange of locality. Now, by choosing g lbs. to ; the unit of mass, we effect this object. For, t W be the weight of a certain body in one cality, say the earth's equator, and g the velocity cquired per second there by falling bodies; also, et W' be the weight of the same body in another part of the universe, say at the sun's equator, and g' the velocity acquired per second by bodies falling there. Then g is the effect per second produced by the force W, acting on the body in question, and g' the corresponding effect produced by the force W' acting on the same body. Hence, by the third law of motion, we have,

$$g : g' :: W : W'.$$

$$\text{And } \therefore \frac{W}{g} = \frac{W'}{g'}.$$

Hence it appears, that, though W and g both vary with the locality, $\frac{W}{g}$ does not, and therefore the above estimation of the mass of a body by the formula $\frac{W}{g}$ is not subject to the difficulty we are considering.

It is to be remembered, however, that we have no real need to consider any besides the common unit of weight in estimating mass, because we may conceive all the matter we are concerned with to be weighed in one particular locality, say at the earth's equator.

Of the Dynamical Effect of Force.—By the *Dynamical Effect* of a force, we mean the effect with reference to its time of action; thus a force acting for one second produces a certain dynamical effect, the same force acting for 2 seconds produces twice as great a dynamical effect, for 3 seconds 3 times as great, and so on; so that, in general, we find the dynamical effect produced in t seconds, by multiplying the dynamical effect produced in one second by t.

The greater the force is, the greater of course is the dynamical effect in proportion. If, then, we assume the dynamical effect produced in a second, by a force of 1 lb., to be unity, the dynamical effect produced in a second by a force of P lbs. will be P; and therefore, by what was stated before, the dynamical effect produced by a force of P lbs. in t seconds, will be t times as much, that is, $P t$.

Assuming, then, the unit of dynamical effect to be the effect produced by a force of 1 lb. in 1 second, it follows that the dynamical effect produced by a force of P lbs. in t seconds, will be the force multiplied by the time, or $P t$.

Momentum.—The word *Momentum* is used in exactly the same sense, in the Mechanical Treatises, as the term Dynamical Effect, just explained; the momentum produced by the force P in the time t, is therefore $P t$.

PROPOSITION V.

To find the Dynamical Effect or Momentum, in terms of the velocity.

Let P be the force, t the time during which it acts, W the weight of the body it acts upon, and v the velocity it produces in the time t; then, by the preceding proposition, we have,

$$v = \frac{P}{W} g t, \text{ and } \therefore \ P t = \frac{W}{g} v.$$

But Pt, as we have shown, is the dynamical effect or momentum; hence,

$$\text{the momentum} = \frac{W}{g} v, \text{ or } Mv, \ M \text{ being the mass.}$$

It appears, then, that the dynamical effect or momentum produced by a force, acting on a body, is found by multiplying the mass of the body $\left(\frac{W}{g}\right)$ by the velocity produced by the force; and this is always true, whatever may be the time during which the force acts.

Corollary 1.—v, which here denotes the velocity *produced by the force*, may or may not be the actual velocity with which the body is moving, because the body may or may not have had a motion previous to the action of the force. If we suppose that the body had a velocity v before the force began to act upon it, and that this velocity is changed into v' by the action of the force, then the velocity produced by the force is not v, but $v' - v$. This follows from the first and second laws of motion.

If therefore the force, by its action on the body, changes the velocity from v to v', the dynamical effect is,

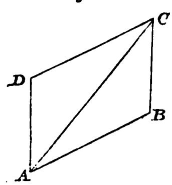
$$\frac{W}{g}(v'-v).$$

We must, therefore, enunciate our rule by saying, that the dynamical effect produced by a force acting on a body, is found by multiplying the *alteration* of velocity produced by the force, by the mass of the body.

Corollary 2.—In all that we have just said, the force and the velocities are supposed to have the same directions, that is, the motion is in all cases rectilineal. If this be not the case, we must define more accurately what we mean by the term *alteration of velocity*.

Suppose that AB, fig. 252, represents in magnitude and direction the velocity which the body had before the action of the force, and that AC represents the velocity after the action of the force, an alteration having been produced by the force in the direction, as well as in the magnitude of the velocity.

Fig. 252.



Complete the parallelogram $ABCD$. Then the velocity AC is equivalent to the two velocities AB and AD together, by the Parallelogram of Velocities. It appears, therefore, that, in addition to the original velocity AB, the force has produced the velocity AD; for, by the first law of motion, the body would always move with the same velocity AB, if the force did not act; but, by the action of the force, the velocity is changed from AB to

AC, or, what is the same thing, the velocity is changed from AB to AB and AD together; the force, therefore, produces the velocity AD in addition to the original velocity AB.

AD is, therefore, the alteration produced in the velocity by the force; and, therefore, the dynamical effect of the force is

$$\frac{W}{g} \times \text{velocity represented by } AD.$$

Proposition VI.

To find the Dynamical Effect or Momentum produced by a variable force.

In the preceding proposition, we have supposed P to be a constant force of so many lbs., neither increasing nor diminishing during its action on the body; we shall now consider the case of a force which varies during its action, either increasing or diminishing. Let the original velocity of the body be v, and suppose a varying force to change this velocity to v' in the following manner:—

Let the force be at first P_1, then P_2, then P_3, then P_4, thus varying from one magnitude to another; let P_1 alter the velocity from v to v_1, P_2 from v_1 to v_2, P_3 from v_2 to v_3, and finally, let P_4 alter the velocity from v_3 to v'; then, by Corollary 1 of the preceding Proposition,

$$\text{the momentum produced by } P_1 = \frac{W}{g}(v_1 - v),$$

$$\text{the momentum produced by } P_2 = \frac{W}{g}(v_2 - v_1),$$

$$\text{the momentum produced by } P_3 = \frac{W}{g}(v_3 - v_2),$$

$$\text{the momentum produced by } P_4 = \frac{W}{g}(v' - v_3).$$

If we add all these momenta together, we shall find the total momentum produced by the varying force; hence, the total momentum produced is,

$$\frac{W}{g}(v_1 - v + v_2 - v_1 + v_3 - v_2 + v' - v_3) \text{ or } \frac{W}{g}(v' - v).$$

It appears, therefore, that the dynamical effect, or momentum, produced by a varying force, is found by the same rule as that produced by a constant force; namely, multiply the total alteration of velocity produced by the force, which is $v' - v$, by the mass of the body $\frac{W}{g}$, and the result will be the total dynamical effect.

This is an important conclusion, for it follows from it that, in finding the dynamical effect produced by a force, we need not inquire whether the force is variable or constant; all we have to do being to find the difference of the velocities which the body has before and after the action of the force. Observe, however, that we suppose the motion to be rectilineal here. If not, we must proceed as in Corollary 2.

Proposition VII.

The alteration of velocity produced in a given time being known, to find the force which acts on the body.

Let v be the velocity of a body at any time t, and v' its velocity at some subsequent time t. By

speaking thus, we mean to indicate instants of time as defined by reference to some Zero;—thus, suppose 12 o'clock to-day to be taken as the Zero from which we reckon time, and assuming seconds as our units, we may say, that a certain event happened at the time 600, meaning thereby that it happened at 600 seconds, or 10 minutes, past 12 o'clock. In the particular case we are considering, when we say that v is the velocity at the time t, we mean, that the body is moving with the velocity v at t seconds (or $\frac{t}{60}$ minutes) past 12 o'clock; and that afterwards, when it is t' seconds past 12 o'clock, (t' being greater than t,) the body is moving with the velocity v'.

Now, if the velocity changes from v to v' in the interval of time $t'-t$, as we suppose it to do, there must be some force acting to produce such a change; let P be that force; then, by the third law of motion,

$$P = \frac{W}{g} \times \text{velocity produced by } P \text{ in 1 second.}$$

Now, $v'-v$ is the velocity produced by P in $t'-t$ seconds, for in that time P changes the velocity from v to v'; therefore, $\frac{v'-v}{t'-t}$ will be the velocity produced in 1 second.*

Hence we have,

$$P = \frac{W}{g} \frac{v'-v}{t'-t}.$$

* For, vel. produced in $t'-t$ seconds : vel. produced in 1 second :: $t'-t$: 1.

$\therefore$ *vel. produced in 1 second* $= \frac{\text{vel. produced in } t'-t \text{ seconds}}{t'-t}$.

In this reasoning we evidently assume that P is a constant force during the time $t'-t$; of this, however, we cannot be sure in general; but if we suppose $t'-t$ to be a very small interval of time, we may always suppose, without material error, that P is constant during it; for however P may vary in an interval of time which is not small, it is manifest that its variation must be very small in a very small interval. If, therefore, we assume $t'-t$ to be an extremely small interval of time, (say the millionth part of a second, or if necessary, the millionth part of the millionth part of the millionth part of a second,) we commit no error worth taking account of when we assume that P is a constant force, neither increasing nor diminishing, during the interval $t'-t$.

On this supposition, then, the force is found by the formula,

$$P=\frac{W}{g}\,\frac{v'-v}{t'-t},$$

which may be stated as a Rule in the following manner.

To determine what force is acting upon a body at any time t, find the velocity of the body at the time t, and the velocity at a subsequent time t', the interval between t and t' being extremely small; then, if we multiply the mass of the body $\left(\frac{W}{g}\right)$ by the fraction $\frac{v'-v}{t'-t}$, we shall obtain the force required.

Observation.—The most rapidly changing motion mathematicians have to deal with is that of the vibrations which constitute light. In this case, by assuming $t'-t$ to be the millionth part of the

millionth part of a second, we should commit serious error, inasmuch as the forces which produce these vibrations increase and diminish 500 times (in round numbers) during that inconceivably minute interval. If, however, we assume $t' - t$ to be the millionth part of the millionth part of the millionth part of a second, no perceptible error will result from the above formula for P.

In the case of falling bodies we need not assume $t' - t$ to be smaller than the thousandth part of a second. In the case of the motion of the planets round the sun, we might assume $t' - t$ to be so great as an hour, without any error.

Thus, the requisite degree of smallness of $t' - t$ depends upon the nature of the motion we are considering; the quicker the forces vary, the smaller must $t' - t$ be assumed. $t' - t$, and $v' - v$, are technically called *little differences* or *differentials*; but on this point we shall say nothing, as it would lead us into the mysteries of the Differential Calculus.

Proposition VIII.

To find an analogous formula for the velocity of a body.

Let s be the space the body describes in the time t, s' that described in the time t'. By this we mean, that, if AB (fig. 253) be the line along which the body moves, and if $AP = s$, and $AP' = s'$; then P is the point the body has arrived at, at the time t, and P', at the time t';

Fig. 253.

so that PP', or $s'-s$, is the space the body moves over in the interval $t'-t$.

We cannot know whether the velocity of the body is constant or variable; but, however this may be, we may, as in the preceding proposition, assume the interval $t'-t$ to be so small, that we may regard the velocity as invariable while the body moves from P to P'. This being the case, if v denote the velocity, we find it (by Prop. I.) by dividing the space $(s'-s)$ by the time of describing it $(t'-t)$. We have, therefore,

$$v = \frac{s'-s}{t'-t};$$

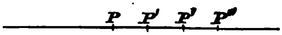
which is the formula required.

Proposition IX.

To explain how the space described in a given time may be found geometrically, when the velocity varies according to a given law.

Let fig. 254 represent the line along which the body moves, and let P, P', P'', &c. represent its successive positions at the end of the times t, t', t'', &c., so that it describes the spaces PP', $P'P''$, $P''P'''$, &c. in the intervals, $t'-t$, $t''-t'$, $t'''-t''$, &c. respectively. Let v, v', v'', &c. be the velocities with which the body describes the spaces PP', $P'P''$, $P''P'''$, &c. respectively; then, by Prop. I. we have,

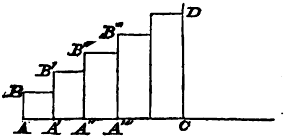
Fig. 254.



$$PP' = v(t'-t),\ P'P'' = v'(t''-t'),$$
$$P''P''' = v''(t'''-t''),\ \&c.$$

Now, in fig. 255, let us draw a line AC, and take every unit of that line to represent a unit of time. This is perfectly allowable; for all that we do is to represent one quantity by another, so far as mere numerical magnitude is concerned, and no further; it is, in fact, as legitimate a representation as that of forces by lines in Statics. This being understood, let us take AA', $A'A''$, $A''A'''$, &c. respectively, to represent the intervals $t'-t$, $t''-t'$, $t'''-t''$, &c. Also, let us draw the perpendiculars AB, $A'B'$, $A''B''$, &c. and make them of proper lengths to represent the velocities v, v', v'', &c. respectively; and complete the rectangles BA', $B'A''$, $B''A'''$, &c.

Fig. 255.



Then the following equations are true *numerically*, viz.:—

$$* \text{ rect. } BA' = AB \times AA' = v(t'-t) = PP'.$$

$$\text{rect. } B'A'' = A'B' \times A'A'' = v'(t''-t') = P'P''.$$

$$\text{rect. } B''A''' = A''B'' \times A''A''' = v''(t'''-t'') = P''P'''.$$

&c. &c. &c.

Hence, by addition, we find that the sum of these rectangles is equal to the sum of the spaces described in the intervals $t'-t$, $t''-t'$, &c. From this it follows, that, if we take on a line AC

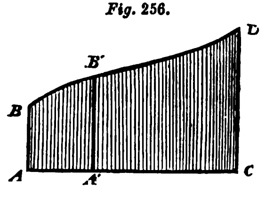
* By the expression, rect. BA', we mean the number of units of surface, *i. e. square units*, in the rectangular surface BA'. By a well-known theorem, this number is found by multiplying the *number of units* in AA' by the number in AB.

the portions AA', $A'A''$, $A''A'''$, &c. to represent successive intervals of time during which, respectively, a body is moving with velocities represented by the perpendicular lines AB, $A'B'$, $A''B''$, &c., the space described in any time, say AC, will be represented by the sum of the rectangles formed, as in the figure, upon AC.

The reader will understand that this mode of representation is purely numerical; thus, for example, suppose that we agree that every inch of AC shall represent a second of time, and every perpendicular inch a foot per second of velocity; then, all that we mean by saying that AC represents the time, AB, $A'B'$, $A''B''$, &c. the successive velocities, and the sum of the rectangles the space described, is this,—that there are as many *seconds* in the time as there are *inches* in AC, as many *feet per second* in the successive velocities as there are *inches* in AB, $A'B'$, $A''B''$, &c., and as many feet in the space described as there are *square inches* in the sum of the rectangles.

Corollary.—If the intervals AA', $A'A''$, $A''A'''$, &c. be extremely small, the points B, B', B'', &c. will lie so close together, that the broken line forming the upper boundary of the rectangles will become a continuous curve, as is shown in fig. 256. In the case represented by this figure, there is a *gradual* instead of an *abrupt* change of velocity as the body moves on, and the curve BD shows the law according to which the velocity varies; *i.e.* if the perpendicular line A′ B′ be drawn to the curve

Fig. 256.



from *any* point A' of AC, it shows the velocity with which the body is moving at the end of the time represented by AA'. The sum of the rectangles on AA' in this case is the surface included between AB, $A'B'$, AA', and the curve BB'.

Hence the following rule. When the velocity varies gradually, draw a curve BD, such that the perpendicular $A'B'$ drawn to the curve from any point A' of AC, shall always represent the velocity with which the body is moving at the end of the time AA'; then the number of square units in the curvilinear area $ABB'A'$ will be the number of feet the body has described in that time along its line of motion, fig. 254.

This method of construction is due to Newton.

EXAMPLES OF THE PRECEDING PROPOSITIONS.

Ex. 1.—A train weighing 100 tons, moving at the rate of 30 miles* per hour, is brought to rest in one minute by the action of a force P; find P.

Ex. 2.—If the train be brought to rest in one second, what is P?

Ex. 3.—A train weighing 100 tons is set in motion by a pressure P, and 10 minutes after starting it is moving at the rate of 40 miles per hour; what is P?

Ex. 4.—Same case, supposing that P is 1 ton; find with what velocity the train is moving half-an-hour after starting.

Ex. 5.—What is the *rate of acceleration* produced by a force of 10 lbs. acting on a weight of 1 ounce?

* Observe, in all these examples, that *time* must be reduced to *seconds*, and *space* to *feet*.

Ex. 6.—What is the *mass* of 1610 lbs.?

Ex. 7.—What is the *dynamical effect* of a force of 1 ton, acting for 1 hour?

Ex. 8.—A body weighing 1 cwt. is moving at the rate of 30 yards per minute, what is its *momentum?*

Ex. 9.—The velocity of a body weighing 100 lbs. is changed from 10 feet per second to 12 feet per second. What is the dynamical effect produced?

Ex. 10.—The same body has its velocity changed from 10 feet per second northward, to 10 feet per second eastward; what is the dynamical effect produced?

N.B. See Cor. 2, Prop. V.

Ex. 11.—The same body has its velocity changed from 10 feet per second northward, to 10 feet per second southward; find the dynamical effect.

Ex. 12.—If the change be from northward to north-eastward, find the dynamical effect.

Ex. 13.—In Ex. 10 the change is produced by the action of a force P in 10 seconds; find P in magnitude and direction.

N.B. See Second Law of Motion.

Ex. 14.—Do the same in Ex. 12.

CHAPTER III.

UNIFORMLY VARIED MOTION.

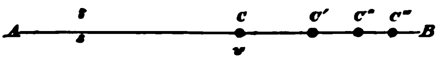
THE motion produced by the action of an invariable force, acting always in the same direction, is not a uniform motion, because, by the "*First Law*," the velocity continues constant only when there is no force in action. The motion in question is of that kind which is called "*uniformly accelerated*," because, by the "*Second Law*," the force will increase the velocity uniformly, that is, it will add, in each second, the same amount of velocity, to the previous velocity. If, however, the force acts in the opposite direction to that in which the body is moving, it will *diminish* the velocity uniformly, by continually subtracting the same amount each second. In this case, the motion is said to be "*uniformly retarded*." There is, however, no essential difference between the two kinds of motion, and they may be both described as "*uniformly varied motion*." This kind of motion, therefore, is that which results from the action of an invariable force acting always in the same direction. Next to "*uniform motion*," it is the most important to be considered, and we now proceed to investigate the formulæ by which it is determined.

Proposition X.

To prove the formulæ for Uniformly Varied Motion, by means of Newton's Construction.

Let W be the weight of the body moving, and P the force acting on it, in the direction AB, fig. 257; let AC be the distance the body describes

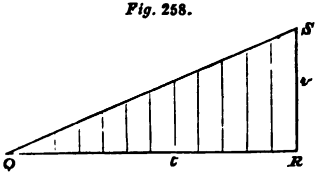
Fig. 257.



by the action of the force in any time t, and let s denote AC; also, let v be the velocity the body has at C. We suppose the motion to begin at A, the body being at rest previous to the action of the force.

Take QR to represent t graphically, (see Prop. IX.) and let perpendiculars to QR represent the velocities at different times, which, since they increase uniformly, will be bounded by a straight line QS, RS representing v, (see Prop. IX.) Now, by the Proposition just quoted, the area of the triangle QSR will represent s; but this area is $\frac{1}{2} QR \times RS$, or $\frac{1}{2} t \times v$; hence we find,

Fig. 258.



$$s = \tfrac{1}{2} v t \quad \ldots\ldots\ldots\ldots \quad (1)$$

Now, if f denote the rate of acceleration, we have (see Props. III. and IV.),

$$f = \frac{P}{W} g, \text{ or } \frac{P}{M} \dots\dots (2)$$

and $v = ft \dots\dots\dots\dots (3)$

Hence (1) becomes, substituting for v,

$$s = \tfrac{1}{2} f t^2 \dots\dots\dots\dots (4)$$

Again, by (3) we have,

$$v^2 = f^2 t^2 = 2f \times \tfrac{1}{2} f t^2;$$

or by (4),

$$v^2 = 2fs \dots\dots\dots\dots (5)$$

(3), (4), and (5), are the formulæ employed in determining cases of uniformly varied motion.

Corollary 1.—The equation (1) shows that the space s is *half* what it would have been had the velocity continued equal to v during the whole motion from A to C. This is sometimes expressed by saying, that the space described from rest, in uniformly accelerated motion, is always half the space due to the last acquired velocity.

Corollary 2.—For falling bodies $f = g$, (32.2), and therefore,

$$v = 32.2\,t$$
$$s = 16.1\,t^2$$
$$v^2 = 64.4\,s.$$

Proposition XI.

To prove the formulæ (4) *for uniformly varied motion, by the method explained in Prop. VII.*

Let C', fig. 257, be the place of the body at any time t' subsequent to the time t; then $CC' = s' - s$, and the time of describing CC' is

$t'-t$; also, v and v' being the velocities at C and C' respectively, we have,

$$v=ft, \quad v'=ft' \quad \ldots\ldots (1)$$

Now, let us assume, as in Prop. VII., that the interval $t'-t$ is extremely small, so small, in fact, that v and v' do not sensibly differ from each other. On this supposition, we may say that CC' is described *uniformly* with either the velocity v or v', or, what is nearer the truth, with the *half-way* or *mean** velocity $\frac{1}{2}(v+v')$. CC' therefore being described in the time $t'-t$, uniformly, with the velocity $\frac{1}{2}(v+v')$, we have, by Prop. I.,

$$\begin{aligned} CC' &= \tfrac{1}{2}(v+v')(t'-t) \\ &= \tfrac{1}{2}f(t'+t)(t'-t) \text{ by (1) above,} \end{aligned}$$

$$\text{or } CC' = \tfrac{1}{2}f(t'^2-t^2).$$

In like manner, if t'', t''', &c. be the times when the body gets to C'', C''', &c. (fig. 257,) and if we suppose the intervals $t''-t'$, $t'''-t''$, &c. extremely small, we may show that

$$C'C'' = \tfrac{1}{2}f(t''^2-t'^2)$$

$$C''C''' = \tfrac{1}{2}f(t'''^2-t''^2),$$

and so on for any number of intervals.

Hence, by addition, observing that

$$CC'+C'C''+C''C''' = CC''',$$

$$\text{and } t'^2-t^2+t''^2-t'^2+t'''^2-t''^2 = t'''^2-t^2,$$

we find, $$CC''' = \tfrac{1}{2}f(t'''^2-t^2).$$

* $\frac{1}{2}(v'+v)$ exceeds v just as much as it falls short of v', as is manifest by simple subtraction; in other words, $\frac{1}{2}(v'+v)$ lies *half-way*, in magnitude, between v and v'; it is, therefore, called the *mean*, or *half-way* velocity between v and v'.

This must manifestly be the result whatever the number of intervals be; in other words, we shall always find, that the space described in any interval of time, small or great, is equal to

$$\tfrac{1}{2}f \,\{(\text{time at which interval commences})^2 - (\text{time at which it ends})^2\}.$$

Now s is the space described in the interval commencing with the time 0, and ending with the time t; wherefore we have

$$s = \tfrac{1}{2}f(t^2 - 0^2),$$

$$\text{or, } s = \tfrac{1}{2}ft^2,$$

which is the formula to be proved.

Observation.—The proof here given is an exemplication of a process of reasoning, which is of very great importance in Natural Philosophy; and it is chiefly on this account that it is given. It may be objected, however, that it is only an approximate proof, and subject to error. This objection leads us to the following explanation.

Degree of accuracy of the proof just given.—We have assumed that the body describes CC' with the mean velocity between v and v'; in so doing we may possibly commit some error in finding CC', but, by taking $t' - t$ sufficiently small, we may obviously make that error less than any specified fraction of CC'. Suppose, then, that the error is less than $\frac{1}{1000000}\,CC'$, and that the same is true also as regards $C'C''$, $C''C'''$, &c. If so, the error committed in finding $CC' + C'C'' + C''C''' +$ &c., is less than $\frac{1}{1000000}(CC' + C'C'' + C''C''' +$ &c.),

and this is true, no matter what the number of intervals be. Wherefore, when we assert that,

$$s = \tfrac{1}{2} f t^2,$$

the error is less than $\frac{1}{1000000} s$.

Now we might, if we pleased, have assumed the intervals so small, that the error should have been less than $\frac{1}{1000000000000} s$, or any other minute fraction of s however small. It follows, therefore, that the equation

$$s = \tfrac{1}{2} f t^2$$

is subject to an error less than the smallest fraction of s than can be specified. But this can be said only of an error Zero; wherefore the error is Zero. Thus it appears that the above proof, so far from being a mere approximation, is rigidly accurate.

Proposition XII.

To determine the formulæ for uniformly varied motion, when the body has an initial velocity.

Let us suppose that the body, instead of starting from rest, has a velocity u to begin with, or, as it is said, an *initial* velocity u. Then the force will generate velocity at the rate of f per second, as before, and thus in any number of seconds (t), there will be a velocity ft *added* to u, if the force act the *same way* as u, or *subtracted*, if the *contrary way*. We have, therefore,

$v = u + ft$ . . . (1) on former supposition,

or, $v = u - ft$ . . . (2) on latter.

Again, in virtue of the initial velocity u, the

body will describe a space ut in the time t, and, in virtue of the action of the force, a space $\frac{1}{2}ft^2$. Wherefore, by the Second Law of Motion, the space actually described will be the *sum* of these two spaces, when the force acts the *same way* as u, and the *difference* when the *contrary way*. We have, therefore,

$s = ut + \frac{1}{2}ft^2$ . . . (3) on former supposition,
$s = ut - \frac{1}{2}ft^2$ . . . (4) on latter.

Lastly, squaring (1), we find,

$$\begin{aligned} v^2 &= u^2 + 2uft + f^2t^2 \\ &= u^2 + 2f(ut + \tfrac{1}{2}ft^2) \\ &= u^2 + 2fs, \text{ by (3).} \end{aligned}$$

Or, if we proceed similarly with regard to (2), we find,

$$\begin{aligned} v^2 &= u^2 - 2f(ut - \tfrac{1}{2}ft^2) \\ &= u^2 - 2fs, \text{ by (4).} \end{aligned}$$

Thus we have,

$v^2 = u^2 + 2fs$ . . (5), force acting same way as u;
$v^2 = u^2 - 2fs$ . . (6), force acting the contrary way.

These are the formulæ for uniformly varied motion, when there is an initial velocity. They may be regarded as all included in the three equations (1), (3), and (5), by giving proper signs to f, u, and s. Thus, (4) is the same thing as (3), only the sign of f is changed, because the direction of f is reversed.

Corollary.—For falling bodies the formulæ become,

$$\begin{aligned} v &= u + 32.2 \times t, \\ s &= ut + 16.1 \times t, \\ v^2 &= u^2 + 64.4 \times s. \end{aligned}$$

But it is to be remembered that u and s here are supposed to be *downward* in direction; if, therefore, u or s be *upward* in direction, the proper change of sign must be made. This will be understood better by the examples which follow, than by any general statement.

EXAMPLES OF UNIFORMLY VARIED MOTION.

Ex. 1.—A stone is let fall; how far does it fall in 10 seconds, and what velocity does it acquire in 10 seconds?

Here use the formulæ $s = 16.1 \times t^2$, and $v = 32.2 \times t$, putting $t = 10$.

Ex. 2.—The stone, instead of being let fall simply, is projected (or thrown) downwards with a velocity 20; how far does it fall, and what velocity does it acquire, in 10 seconds?

Here use the formulæ $s = u\,t + 16.1 \times t^2$, and $v = u + 32.2 \times t$, putting $u = 20$, and $t = 10$.

Ex. 3.—In Ex. 2, suppose that the stone is projected *upwards*, instead of downwards; find the space and velocity.

Here u will be *against* gravity, and therefore, in estimating the space descended by the stone in any time, we must *subtract* $u\,t$, instead of adding. We have, therefore,

$$s = -u\,t + 16.1 \times t^2, \text{ and } v = -u + 32.2 \times t,$$

in which, put $u = 20$, and $t = 10$.

Ex. 4.—In Ex. 3, how high will the stone have gone in 1 second?

Here, putting $t = 1$ in the formula, $s = -u\,t + 16.1 \times t^2$, (and $u = 20$,) we find,

$$s = -20 + 16.1.$$

Wherefore, the space described is 20 *upwards*, and 16.1 *downwards*, i.e. 3.9 *upwards*.

N.B. The answer in this case is, $s = -3.9$; and here the meaning of the negative sign should be particularly noted. Originally s was regarded as a *downward* space; but here we find s to be an *upward* space, and this is indicated algebraically by the negative sign. The fact is, the negative sign always indicates *reversed direction*, or direction measured contrary to what we originally supposed.

Ex. 5.—How long will the stone continue to ascend, if it be thrown upwards with a velocity of 100?

Here $v = -100 + 16.1 \times t$; and it is clear that the stone will continue its upward motion as long as the velocity due to gravity ($16.1 \times t$) is less than the velocity of projection* (100); but as t increases, the former velocity will increase, and (by the formula) at last become equal to 100; in which case v will be reduced to zero, that is, the stone will cease moving; only for an instant, however, for gravity will immediately begin to produce a downward motion. Hence we shall find the time during which the stone continues to ascend, by putting

$$16.1 \times t = 100,$$

$$\text{or } t = \frac{100}{16.1} = \frac{25}{4}, \text{ nearly, } = 6\tfrac{1}{4} \text{ seconds.}$$

Ex. 6.—In the same case, how high will the stone ascend?

* The *velocity of projection* means the velocity communicated to the body at starting, whether by the hand, or by a blow, or by the *force of gunpowder*, or otherwise.

In $6\frac{1}{4}$ seconds the stone will ascend a space given by the formula $s = -100\,t + 16.1 \times t^2$, putting $t = 6\frac{1}{4}$; but the proper formula for this case is,

$$v^2 = u^2 + 64.4 \times s.$$

Here put $u = 100$, and we shall find the velocity v corresponding to any space s; but $v = 0$, when the stone is at its highest elevation, as explained in Ex. 4; wherefore,

$$0 = (100)^2 + 64.4 \times s = 10{,}000 + 64.4 \times s;$$

$$\text{whence } s = -\frac{10{,}000}{64.4} = -156 \text{ feet nearly.}$$

Here the negative sign, as before, means that s is an *upward* space.

Ex. 7.—How high will a stone, thrown upwards with a velocity 10, ascend?

Ex. 8.—With what velocity must a stone be thrown upwards, that it may ascend a quarter of a mile?

Ex. 9.—Where will the stone be (Ex. 8) at the end of one minute?

Ex. 10.—A stone is let fall from the top of a pillar, and it is observed to strike the ground in a second and a half; how high is the pillar?

Ex. 11.—A stone is let fall from a cliff 128 feet high; in what time will it reach the bottom?

Ex. 12.—If (in Ex. 11) the stone be projected downwards with a velocity 32, in what time will it reach the bottom?

Here we have

$$128 = s = 32\,t + 16.1\,t^2.$$

Or, very nearly, omitting decimals, and dividing by 16,

$$t^2 + 2t = 8.$$

Whence,
$$t^2 + 2t + 1 = 9$$
$$t + 1 = \pm 3$$
$$t = 2, \text{ or } -4.$$

Here we obtain two answers, which often happens in problems of this kind. Of course, $t = 2$ seconds is the answer we are seeking for; the other answer, $t = -4$, corresponds to a *previous* motion of the stone *ascending;* for it will be found, that, if the stone were projected upwards from the bottom 4 seconds *before* the instant zero, (*i.e.* $t = 0$,) with a velocity 96 (omitting decimals), it would arrive at the top of the cliff, in its descending motion, at the instant zero, and with a velocity 32. Now this corresponds to the data of the problem; for all that is therein given amounts to this, that the stone leaves the top of the cliff at the instant zero with a descending velocity 32; and the thing sought is, the time, *i.e.* the value of t, when the stone *will be,* or *was* at the bottom; *positive* values of t corresponding to *future,* and *negative* to *past* time. So the answer given by the equation is necessarily double, giving a future and a past time.

N.B. When any difficulty occurs about double answers, the simplest method is to test each answer by substitution, and so determine whether it agrees with the suppositions of the problem.

Ex. 13.—A stone is let fall from the top of a cliff, and the interval that elapses before the sound

of its striking the bottom is heard, is $5\frac{1}{2}$ seconds; to find the height of the cliff.

Let s be the height, and t the time the stone takes to fall down; then $5\frac{1}{2}-t$ will be the time the sound takes to come up. Sound, in ordinary weather, and in open air, travels uniformly at about 1120 feet per second. We have, therefore,

$$s=16.1\,t^2=16\,t^2, \text{ omitting decimal;}$$

also, for the ascending sound,

$$s=1120\,(5\tfrac{1}{2}-t).$$

Wherefore,

$$\begin{aligned}16\,t^2&=1120\,(5\tfrac{1}{2}-t)\\ t^2&=70\,(5\tfrac{1}{2}-t)\\ t^2+70\,t&=385\\ t^2+70\,t+1225&=1610.\end{aligned}$$

Whence, extracting the square root, and omitting decimals, we find,

$$t+35=\pm 40 \text{ nearly.}$$

$$t=5, \text{ or } -75.$$

The negative answer may be explained as in the former example; the positive answer, $t=5$, is that which we are seeking; and since $s=16\,t^2$, the height required is 16×25, or 400 feet.

If we had omitted the consideration of the velocity of sound, the answer would have been,

$$s=16\,(5\tfrac{1}{2})^2=16\times 30 \text{ nearly,}$$

$$\text{or } s=480.$$

Ex. 14.—A stone is projected upwards with a

velocity 64; find when it is 48 feet above the ground.

$$\text{Here } s = 48 \text{ feet } \textit{upwards} = -48,$$

$$u = 64 \text{ feet } \textit{upwards} = -64.$$

Wherefore, the equation, $s = ut + 16.1t^2$, becomes, omitting decimals,

$$-48 = -64t + 16t^2.$$

Whence,
$$t^2 - 4t = -3$$
$$t^2 - 4t + 4 = 1$$
$$t - 2 = \pm 1$$
$$t = 3, \text{ or } 1.$$

The double answer here has an obvious meaning; for the stone rises 48 feet in 1 second, and, in its descent, it is 48 feet above the ground again at the time $t = 3$. This may be verified easily by substitution.

Ex. 15.—A stone let fall from a pillar, is observed to fall down the last half of the pillar in 1 second; find the height of the pillar.

Let s be the height required, and t the time of falling down it; then

$$s = 16.1 \times t^2.$$

Also, since $t - 1$ is the time of falling down the height $\frac{s}{2}$, we have,

$$\frac{s}{2} = 16.1 \times (t - 1)^2.$$

Wherefore, $t^2 = 2(t-1)^2$,

which gives t, and thence s.

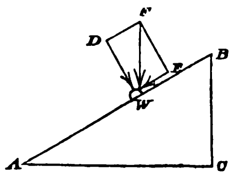
Ex. 16.—Show that a stone thrown up comes down again to the ground with the velocity of projection, and that the time of ascent is equal to that of descent.

Proposition XIII.

To determine the motion of a body down a smooth inclined plane.

Let $A\,B$ be the inclined plane, fig. 259, $A\,C$ its base (horizontal), $B\,C$ its height (vertical), and let us, for brevity, put $A\,C=b$, $B\,C=h$, $A\,B=l$. Let W be the weight of the body moving down $A\,B$, which represent by the vertical arrow $F\,W$; and resolve $F\,W$ into the two forces, $E\,W$ along the plane, and $D\,W$ at right angles to it. Thus, as in Statics, (p. 294,) we have,

Fig. 259.



$$E\,W : F\,W :: h : l.$$

Wherefore, $$E\,W=F\,W\,\frac{h}{l},$$

$$\text{or, force } E\,W=W\,\frac{h}{l}.$$

Now the force $D\,W$, being at right angles to the direction along which W moves, cannot produce any moving or retarding effect, since the plane is supposed to be perfectly smooth. Wherefore, the force $E\,W$ alone produces motion. It follows, therefore, that the motion of W along AB is the

effect produced by the action of the force $E W$, or, as has been shown,

$$\text{the force } W \frac{h}{l}.$$

Let f be the rate of acceleration which this force produces; then, by Dynamics, Prop. III., putting for P the force just obtained, we find,

$$f = \frac{P}{W} g = \frac{h}{l} g \quad . \quad . \quad . \quad . \quad (1)$$

The rate of acceleration being thus obtained, it is clear that the motion of W down the plane is determined, by the formulæ in the preceding propositions.

Ex.—If $h = \frac{1}{4} l$, to find the rate of acceleration of W down the plane.

$$\text{Here } f = \frac{1}{4} g = 4 \text{ nearly.}$$

Wherefore W, in moving down AB, gains velocity at the rate of 4 per second nearly.

Observation. $\frac{h}{l}$ is the sine of the angle of inclination of the plane AB to the horizon, *i.e.* the angle BAC. Wherefore,

$$f = g \sin. BAC.$$

Corollary 1.—*To find the velocity of W when it has descended down the plane, and the time occupied in the motion.*

Let t be the time of motion from B to A, supposing that W is simply let go at B; and let v be

the velocity at A. Then, by the former propositions, we have,

$$v^2 = 2fs = 2fl,$$

or, by (1) present proposition, $v^2 = 2gh$ . . . (2)

This gives the velocity required.

To find t, we have, by the former propositions,

$$s = \tfrac{1}{2} f t^2,$$

$$\text{or, } l = \tfrac{1}{2} g \frac{h}{l} t^2.$$

$$\text{Wherefore, } t^2 = \frac{2l^2}{gh} \quad \ldots\ldots\ldots\ldots \quad (3)$$

This gives the time of motion required.

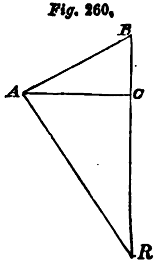
Corollary 2.—*Interpretation of formula* (2) *just obtained.*

The formula (2) is thus interpreted. If W, instead of moving down BA, were allowed to fall freely from B to C, the velocity at C would be given by the formula,

$$v^2 = 2gh,$$

as has been shown in the preceding propositions. Wherefore it appears by the formula (2) that the velocity acquired in going down from B to A along the plane, is just the same as that acquired in falling freely from B to C. In other words, the velocity acquired in going down an inclined plane is that due to the vertical height of the plane.

Fig. 260.



Corollary 3.—*Interpretation of formula* (3) *just obtained.*

In fig. 260, A, B, C denote the same as in fig. 259; and AR is drawn at right angles to A B, to meet B C pro-

duced at R. Wherefore ABR and ABC are similar triangles; and consequently we have,

$$RB : BA :: BA : BC$$

$$\therefore\ RB = \frac{BA^2}{BC} = \frac{l^2}{h}.$$

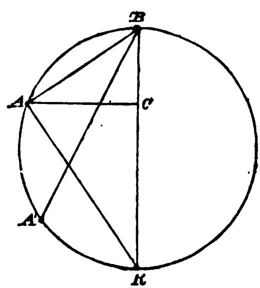
Hence, by formula (3) above, we have,

$$t^2 = \frac{2\,RB}{g}.$$

Now this formula, by the preceding propositions, indicates that t is the time in which a body falling freely descends from B to R. Hence, since t is the time a body takes to move down from B to A, it follows that the time a body takes to go down the inclined plane BA, is the same as the time of falling freely from B to R.

This may be further interpreted, constructively, by describing a circle about the triangle ABR, as in fig. 261. Of course, since BAR is a right angle, BR will be the diameter of that circle: wherefore, we may state the result just obtained, by saying that the time a body takes to go down a chord BA of a vertical circle, supposing that chord to be an inclined plane, is the same as the time a body takes to fall freely down the vertical diameter.

Fig. 261.



Since BA may be *any* chord drawn from B, it *follows from this*, that the time down any other

chord BA' is the same as the time down BA; and, generally, that the times down all chords drawn from B are equal. Observe, that B is the highest point of the circle.

Corollary 4.—To find the plane of shortest descent from a given point B, to a given line EF, fig. 262.

Draw BS vertical, and describe a circle, passing through B, touching EF, and having its centre on BS. (How this is to be done will be explained.) Let A be the point of contact, and draw BA. Then BA is the plane of shortest descent required. For draw any other planes, BH and BG, cutting the circle at A'' and A'. Then the time down BA is equal to the time down BA', by what has been proved; but the time down BA' is shorter of course than the time down BG: wherefore the time down BA is shorter than that down BG. The same may be shown with reference to BH, or any other plane. It follows, consequently, that BA is the plane of shortest descent.

Fig. 262.

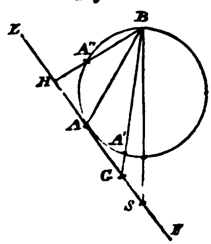
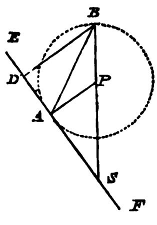


Fig. 263.



To describe the circle here employed, draw BD at right angles to EF, fig. 263, and then draw BA bisecting the angle DBS; which being done, draw AP parallel to DB, and therefore at right angles to EF. Then P is the centre of the required circle; as may be easily shown by the Third Book of Euclid.

Observation.—A great variety of problems may be solved respecting planes of quickest descent after the manner of construction here given; but they are neither of practical importance, nor illustrative of mechanical principles, and therefore we pass them over.

Proposition XIV.

To find the effect of friction in retarding the motion of a body down an inclined plane.

Recurring to page 440, and fig. 259, we have,

$$DW : FW :: AC : AB :: b : l,$$

$$\text{and } \therefore DW = FW\frac{b}{l} = W\frac{b}{l}.$$

But DW being the perpendicular pressure against the plane, we have, by Statics, chap. vii.,

$$\text{force of friction} = \mu DW = \mu W\frac{b}{l}.$$

Wherefore, since friction is a retarding force acting opposite to EW, the actual force on W urging it down the plane is,

$$EW - \mu DW,$$

$$\text{or, } W\frac{h}{l} - \mu W\frac{b}{l} = P \text{ suppose.}$$

Wherefore, if f denote the rate of acceleration, we have, by Dynamics, Prop. III.,

$$f = \frac{P}{W}g = \frac{h - \mu b}{l}g \;\ldots\ldots\; (1)$$

Hence, if t denote the time the body takes to descend from B to A, and v the velocity it has acquired when at A, we have, by Prop. X.,

$$l = \tfrac{1}{2} f t^2 = \tfrac{1}{2} \frac{h - \mu b}{l} g t^2,$$

$$\text{or, } t^2 = \frac{2 l^2}{g(h - \mu b)} \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{Also, } v^2 = 2 f l = 2 (h - \mu b) g \quad . \quad . \quad . \quad (3)$$

(2) and (3) determine t and v.

Corollary.—If the motion be *up* the plane, the friction and gravity are both retarding forces, and therefore we have

$$P = E W + \mu D W$$

$$\text{and } \therefore f = \frac{h + \mu b}{l}.$$

Here f is a retardation.

EXAMPLES OF MOTION ON AN INCLINED PLANE.

In these examples, the formulæ given in Propositions X. and XII. Dynamics, are to be used, together with the values of f just obtained in Propositions XIII. and XIV. Fig. 259 is the one referred to.

Ex. 1.—A body takes 10 seconds to descend from B to A, and requires a velocity equal to $5g$; what is the gradient, *i.e.* what fraction is h of l?

We have generally, by Prop. X.,

$$v = f t,$$

$$\text{but } v = 5\,g, \text{ and } t = 10,$$

$$\therefore f = \tfrac{1}{2} g.$$

But by Prop. XIII. $f = \frac{h}{l} g$;

wherefore, $\frac{1}{2} = \frac{h}{l}$, or $h = \frac{1}{2} l$.

Ex. 2.—If the body be projected (started) from B with a velocity 4 down the plane, and if $h = \frac{1}{2} l$, find the velocity at A, supposing $l = 10$.

Here $f = \frac{1}{2} g$, and, by Prop. XII.,

$$v^2 = 4^2 + 2fl,$$
$$= 16 + 10g = 338;$$
$$\therefore \; v = 18 \text{ nearly}.$$

Ex. 3.—A body is projected from f with a velocity 8 up the plane; how high will it ascend along the plane?

By equation (6), Prop. XII., we have,

$$v = u^2 - 2fs.$$

Now here, $u = 8$, and $f = \frac{1}{2} g = 16$ nearly.

Wherefore, $v^2 = 64 - 32\,s$ nearly.

If in this equation we put

$$64 - 32s = 0, \text{ or } s = 2,$$

we find $v = 0$, whenever the velocity becomes zero, when the body has gone 2 feet up the plane, and therefore it will go 2 feet, and no more, up the plane.

Ex. 4.—With what velocity must the body be started up the plane from A, in order to arrive at B with a velocity 9?

Here, if we put $s = 10$, and $v = 9$, and $f = \frac{1}{2} g$, in equation 6, Prop. XII., we find,

$$81 = u^2 - 322,$$

$$\therefore \; u^2 = 403, \text{ and } \therefore \; u = 20 \text{ nearly.}$$

Wherefore 20 is the required velocity nearly.

Ex. 5.—Find the time the motion up the plane occupies in Ex. 3 and Ex. 4.

In Ex. 3, we have, by formula 2, Prop. XII.,

$$v = u - ft = 8 - 16\,t.$$

Now, here, $v = 0$ when $8 - 16\,t = 0$, or $t = \frac{1}{2}$.

Wherefore $\frac{1}{2}$ second is the time required.

In Ex. 4, we have, similarly,

$$v = u - ft = 20 - 16\,t.$$

But $v = 9$ at B; wherefore,

$$9 = 20 - 16\,t, \text{ or } t = \frac{11}{16}.$$

Ex. 6.—If the plane be rough, $\mu = \frac{1}{3}$, $l = 5$, $h = 4$, and, therefore, $b = 3$, find what time the body takes to go down from B to A.

Here, by (1) Prop. XIV., we have,

$$f = \frac{h - \mu b}{l}\, g$$

$$= \frac{4 - 1}{5} \,.\, 32.2 = 19.3.$$

Wherefore, by the same Proposition,

$$l = \tfrac{1}{2} f t^2, \text{ or } 5 = 9.6 \times t^2, \text{ nearly;}$$

$$\therefore \; t^2 = \sqrt{\frac{5}{9.6}} = \frac{5}{7} \text{ nearly.}$$

Ex. 7.—Supposing that the body is set going up the plane from A, with a velocity 10, and arrives at B in 4 seconds, find μ. ($h=4$, $l=5$, $b=3$).

Here we have, by (4) Prop. XII.,

$$s=ut-\tfrac{1}{2}ft^2.$$

Whence, putting $s=5$, $u=10$, $t=4$, we find,

$$5=40-8f, \text{ and } \therefore f=\frac{35}{8}.$$

But, by (1) Prop. XIV. Cor. we have,

$$f=\frac{h+\mu b}{l}=\frac{4+3\mu}{5}.$$

Wherefore, $\dfrac{35}{8}=\dfrac{4+3\mu}{5}$;

$$\therefore\ 175=32+24\mu.$$

Whence, $\mu=\dfrac{143}{24}=6$ nearly.

Ex. 8.—Supposing $l=10$, and $h=6$, find what value of μ will make $f=4$ for motion up the plane.

By calculation or measurement, we shall find $b=8$. Wherefore, for motion *up* the plane, we have, Prop. XIV. Cor.,

$$4=f=\frac{h+\mu b}{l}=\frac{6+8\mu}{10}.$$

$$\therefore\ \mu=4\tfrac{1}{4}.$$

Ex. 9.—If a body is projected up the plane from A with a velocity 3, supposing $l=10$, $b=6$,

$h=8$, and $\mu=\frac{1}{5}$, find how high the body will ascend up the plane, and how long it will take before it comes back to A.

For the ascent we have,

$$f=\frac{h+\mu b}{l}=\frac{8+2}{10}=1;$$

$$\text{and } v^2=u^2-2fs=9-2s.$$

Wherefore, putting $v=0$, to find s, we have,

$$s=\frac{9}{2}=4\tfrac{1}{2}.$$

This shows how high the body will ascend.

We have also,

$$v=u-ft=3-t,$$

$$\text{or, putting } v=0,\ t=3,$$

whence 3 seconds is the time occupied by the ascent.

For the descent we have,

$$f=\frac{h-\mu b}{l}=\frac{8-2}{10}=\frac{3}{5};$$

and measuring s downwards from the highest point attained by the body, we have,

$$s=\tfrac{1}{2}ft^2=\frac{3}{10}t^2.$$

Now, when the body comes to A, s is $4\frac{1}{2}$, since the whole ascent is $4\frac{1}{2}$. Wherefore, if t be the whole time of descent, we find

$$4\tfrac{1}{2}=\frac{3}{10}t^2, \text{ or } t^2=15;$$

$$\text{and } \therefore\ t=4 \text{ nearly.}$$

Thus the whole time of ascent and descent together is $3 + 4$, or 7 seconds nearly.

Ex. 10.—Find the time of descent down the plane (smooth), when $h = 10$, $b = 100$.

Ex. 11.—Find the same when plane is rough, and $\mu = \frac{1}{20}$.

Ex. 12.—Find, or try to find, the same when $\mu = 1$, and explain the reason of the failure.

Ex. 13.—Same case as in Ex. 10. Find the time of descent when the body is projected from B down AB with a velocity 4.

Ex. 14.—Find the same if plane be rough, and $\mu = \frac{1}{20}$.

Ex. 15.—Same case as in Ex. 12. Find what velocity of projection will be sufficient to make the body just reach the bottom without stopping.

Ex. 16.—Same case. If the velocity of projection be only half what it ought to be for the purpose, how far will the body go down the plane?

Ex. 17.—The body (same case) is projected up the plane from A with a velocity 4; how high will it ascend?

Ex. 18.—Find the smooth plane of shortest descent from a given point to a given circle.

Ex. 19.—Find the smooth plane of shortest descent from a straight line to a circle.

Ex. 20.—Give a geometrical construction for showing the time of descent down a given rough plane.

CHAPTER IV.

SOME CASES OF CURVILINEAL MOTION CONSIDERED—CENTRIFUGAL FORCE—MOTION DOWN A CURVE.

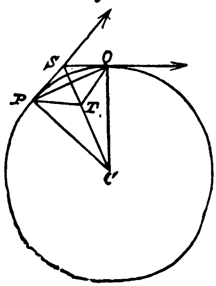
Curvilineal Motion may be investigated by the aid of the second law of motion; but cases of this species of movement, sufficiently simple for a treatise like the present, are very few. The following, however, may be easily managed:—

UNIFORM CIRCULAR MOTION.

This kind of motion takes place when a body describes the circumference of a circle, always with the same unchanging velocity. It is a very important and fundamental case of curvilineal motion, and deserves special attention.

Proposition XV.

Fig. 264.



A body describes a circle (radius r), with a uniform velocity (v), to find the force (F) which produces this motion.

It is clear that there must be a force in action, in this case, for, if not, the body would describe a straight line, not a circle, by the first law of motion.

Let C, fig. 264, be the centre

of the circle, and suppose that the body moves from P to Q in a second. Draw PS and SQ, tangents at P and Q; complete the parallelogram $PSQT$, and draw its diagonal ST, which, produced, will manifestly pass through C.* The body when at P, is moving with a velocity v, in the direction PS, and when at Q, with the same velocity in the direction SQ. Wherefore, since PT is parallel to SQ, and equal to PS, we may take PS and PT to represent the velocity (v) of the body, at P and Q respectively. Now, by Prop. V. Cor. 2, the force (F) which has produced this change of velocity is,

$$F = \frac{W}{g} \times \text{(velocity represented by } ST) \quad . \; . \quad (1).$$

But the triangles PST and PQC are manifestly similar; wherefore,

$$ST : PS :: PQ : PC,$$

$$\text{or, velocity } ST : v :: PQ : r.$$

$$\therefore \text{ velocity } ST = \frac{vPQ}{r} \; . \; . \; . \; . \; . \quad (2).$$

Now I shall suppose that the movement is slow enough to allow our assuming the arc PQ, which is described in a second, to be, practically, of the same length as the chord PQ. This gives chord $PQ = v$, and therefore (2) becomes

$$\text{velocity } ST = \frac{v^2}{r}.$$

* This follows from the fact that the two tangents SQ and SP are, as is well known, always equal.

And therefore, by (1), we find

$$F = \frac{W}{g} \frac{v^2}{r} \quad . \; . \; . \; . \; . \; (3).$$

If, however, the motion is quick, so that PQ is an arc of some length, and therefore chord PQ is decidedly less than arc PQ; we have only to take the millionth part of a second as our time-unit, instead of a second, and suppose that PQ is described in this unit.

But it is clear that, if we diminish our time-unit, we must diminish v, g, and F,* in the same proportion; *i.e.* we must put $\frac{v}{1,000,000}$, $\frac{g}{1,000,000}$, $\frac{F}{1,000,000}$, instead of v, g, and F. Now, making these substitutions in (3), we find

$$F = \frac{W}{g} \frac{v^2}{r} \quad . \; . \; . \; . \; . \; (3).$$

So the formula for F is not affected by this change, and therefore the result is true for all motions, however quick.

Observation.—This value for F is a very important result. It shows that the force which produces uniform circular motion is got by *multiplying* $\frac{W}{g}$ (*i.e.* the *mass* of the moving body) *by the square of the velocity, and dividing by the radius.* Also, since ST produced goes through C, it appears that *this force always acts directly towards the centre.*

* Because v is the space described, g the velocity acquired (by falling body), and F the amount of force brought into play, *in a unit of time.*

Corollary.—Given the *number* of revolutions per second, to find F.

Suppose that W goes round the circle n times in a second. The space described in one revolution by the moving body is $\frac{63}{10}r$, or, more exactly, $2\pi r$, where π is simply used, for brevity, to denote the number 3.14159 (see p. 225): wherefore, the space described in n revolutions is $2\pi n r$. This, then, is the velocity (by definition, p. 382), wherefore $v = 2\pi n r$ in (3); and thus we find,

$$F = \frac{W}{g}\,\frac{4\pi^2 n^2 r^2}{r}.$$

$$\text{Or, } F = \frac{W}{g} 4\pi^2 n^2 r \quad . \; . \; . \; . \quad (4).$$

The number n, *i.e.* the number of revolutions per second, I shall call the *circular velocity*.

OF CENTRIFUGAL FORCE.

If the body be caused to move in a circle, by fastening it to one end P of a string, the other end being fixed to the centre C, and giving it an initial impulse; the tension of the string tending to pull the body *in towards* the centre, will be the force F found in the proposition; *i.e.* the tension of the string will be a force

$$\frac{W}{g}\,\frac{v^2}{r}.$$

Now, if there is this tension on the string, there must of necessity be an equal and opposite reaction on the point C, to which the string is fixed;

i.e. an *outward* force $\frac{W}{g}\,\frac{v^2}{r}$ tending always to pull *C towards the moving body.*

This *outward* pull *on the centre* is called a *centrifugal,* or centre-flying force.

The *inward* pull on *W* exercised by the tension of the string, is called a *centripetal,* or centre-seeking force.

Observe, the *centrifugal force does not act on the body,* but on the centre. The centrifugal force is the outward *reaction* on the centre, or on the string, or whatever it be that forces the body to move in a circle; but it is not a force acting on the body itself.

N.B.—A large amount of error has been put in circulation on the subject of centrifugal force by popular writers, and others who ought to know better. Hegel's celebrated attack on the Theory of Gravitation, and his method of showing that the Planets do not move as Newton asserted, but rather, that they "go along like blessed Gods," is based principally on the error I allude to, which Hegel, not having a "judicious and distinctive head," greedily swallowed.

Here is the common notion of the planetary motions actually received by the bulk of educated men at the present day. A planet has a circular motion about the sun: therefore, a centrifugal force acts upon it tending to drive it outwards from the sun. But the sun attracts the planet, and so produces an equal and opposite inward or centripetal force. Wherefore the two forces, the centripetal and centrifugal, exactly destroy each other; and the consequence is Circular Motion!!

But circular motion is not the consequence of

no force acting on a body. What says the *first law of motion?* When a body in motion is acted on by no forces, it moves *in a straight line, not in a circle.* Thus, according to these erroneous notions, the effect of the circular motion of the planet is to produce centrifugal force; and the effect of the centrifugal force is to destroy the centripetal force; and then no force acts on the body; and then the body moves in a straight line. Thus the result which follows from the *circular* motions of the planets about the sun is, that they all move in *straight lines!!*

The real state of the case is simply this. A body will not move in a circle unless there be a force always acting upon it directly *towards* the centre, namely the force $\frac{W}{g}\frac{v^2}{r}$ found above. The sun by its attraction exerts this force on each planet, and so produces its circular motion. It must be remembered, however, that the motions of the planets about the sun are not exactly circular, and therefore the force exercised is not exactly $\frac{W}{g}\frac{v^2}{r}$.

Examples of Proposition XV.

Ex. 1.—A body, weighing 1 lb., held by a string 10 feet long, is whirled round uniformly at the rate of three revolutions per second; find the tension on the string.

The *circular velocity* (*i.e.* the number of revolutions per second) is, here, 3; wherefore, by the *Cor., Prop.* XV.

$$F = \frac{W}{g}(2\pi)^2 n^2 r$$

$$= \frac{(6.3)^2 (3)^2 10}{32.2} \text{ nearly.}$$

$$= 112 \text{ nearly.}$$

Hence the pull on the string is 112 lbs., or 1 cwt.

Ex. 2.—If the number of revolutions be only 1 per second, find F.

Here $n = 1$, and $F = 12$ lbs. nearly.

Ex. 3.—If the number of revolutions be 5 per minute, find F.

Here the circular velocity is 5 per 60 seconds, or $\frac{1}{12}$th of a revolution per second; *i.e.* $n = \frac{1}{12}$.

$$\therefore\ F = \tfrac{1}{12} \text{ lb. nearly.}$$

Ex. 4.—If, in Ex. 1, the string is only 1 foot long, find F.

Here $n = 3$, $r = 1$, and $\therefore$ $F = 11$ nearly.

Ex. 5.—How fast is the body whirled round if the tension on the string is equal to W $(r = 10)$?

Ex. 6.—How long is the string if the tension is equal to 10 W, $(n = 3)$?

MOTION OF A PROJECTILE.

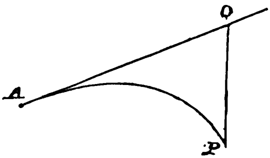
Another case of curvilineal motion that may be easily solved, is that of a body projected with a given velocity in a given direction; the resistance of the air being neglected. The body so projected is called a *projectile*.

Proposition XVI.

To find where a projectile will be at any given time after it has been projected.

Let u denote the velocity of projection, and t the given time. Let AQ, fig. 265, be the direction in which the body is projected from the point A. Take $AQ = ut$, and draw QP downwards equal to $\frac{1}{2}gt^2$; then P is the point where the body will be at the end of the time t.

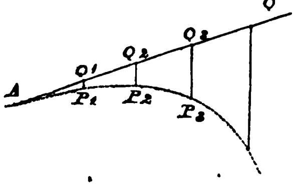
Fig. 265.



This follows immediately from the extended statement of the second law of motion in page 394; for AQ is the space described in the time t, in virtue of the velocity u, and QP the space the body falls in the time t by gravitation. Wherefore, by drawing AQ and then QP, we arrive at the point P, where the body actually is at the end of the time t.

Corollary.—To show the successive positions of the body at the end of each successive second.

Fig. 266.



Take, along AQ, fig. 266, the portions AQ_1, Q_1Q_2, Q_2Q_3, &c., each equal to u; draw, vertically, Q_1P_1, Q_2P_2, Q_3P_3, &c., equal respectively to

$$\tfrac{1}{2}g,\ \tfrac{1}{2}g \times 4,\ \tfrac{1}{2}g \times 9,\ \tfrac{1}{2}g \times 16,\ \&c.$$

Then, it is obvious that P_1, P_2, P_3, &c. will be the positions of the body at successive seconds.

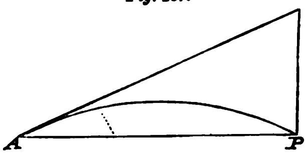
Hence the form of the curve described by the body may be easily drawn. It is called a Parabola.

Proposition XVII.

To find where the body strikes the ground, (which is supposed to be horizontal,) and the time of flight.

AP, in fig. 267, represents the ground, AQ the direction of projection, and QP is vertical. Let the time of motion, or the *time of flight*, as it is called, be t; then, as before,

Fig. 267.



$$AQ = ut, \quad QP = \tfrac{1}{2} g t^2.$$

Now, let the gradient of AQ be one in n feet, (*i.e.* for every n feet you go along AQ you rise one foot,) then $AQ = n\,QP$, and $\therefore$

$$ut = n \tfrac{1}{2} g t^2;$$

$$\text{or, } t = \frac{2u}{ng} \quad \ldots\ldots \quad (1).$$

This gives the time of flight.

To find AP, (the distance at which the projectile strikes the ground,) we have

$$AP^2 = AQ^2 - QP^2$$

$$= AQ^2 \left(1 - \frac{1}{n^2}\right) \because QP = \frac{1}{n} AQ.$$

$$\therefore AP = AQ\sqrt{1 - \frac{1}{n^2}}.$$

$$\text{But } AQ = ut = \frac{2u^2}{ng} \text{ by (1);}$$

$$\therefore AP = \frac{2u^2}{ng}\sqrt{1 - \frac{1}{n^2}} \; . \; . \; . \; . \; (2).$$

This gives AP.

Trigonometrical Solution.—Let the angle QAP, which is called the *angle of projection*, be a; then

$$\sin. a = \frac{QP}{AQ} = \frac{\frac{1}{2}gt^2}{ut} = \frac{1}{2}\frac{g}{u}t;$$

$$\therefore t = \frac{2u}{g}\sin. a \; . \; . \; . \; . \; . \; . \; (3).$$

$$\text{Also, } AP = AQ\cos. a = ut\cos. a.$$

$$\text{And } \therefore \text{ by (3) } AP = \frac{2u^2}{g}\sin. a\cos. a \; . \; . \; . \; . \; (4).$$

Corollary.—$AP = \frac{u^2}{g}\sin. 2a$, by (4). Now, $\sin. 2a$ is greatest when $2a$ is $90°$, *i.e.* when $a = 45°$.

Wherefore, the greatest value of AP, or, as it is called, the greatest *horizontal range*, is obtained by projecting the body at an inclination of $45°$ to the horizon. This is not true when the velocity of projection is great, because then the resistance of the air is considerable, and we have left that out of account altogether.

Ex. 1.—Find t and AP when $n=2$, and $u=64$. Here by (1) and (2) we have,

$$t = \frac{2 \times 64}{2 \times 32} = 2 \text{ seconds, nearly.}$$

$$AP = \frac{2(64)^2}{2 \times 32} \sqrt{1 - \frac{1}{4}} = 110 \text{ feet nearly.}$$

Ex. 2.—Find AP when $n=10$, and $u=100$.

Ex. 3.—Find the greatest elevation the body attains, if it be projected at an angle of 30°, with a velocity of 128.

In this case, resolve the velocity into two velocities, one vertical, and the other horizontal. The vertical will be found by construction, or calculation, to be 64. Now, the horizontal velocity can have no effect as regards the vertical motion; wherefore, the greatest height attained will be got from the equation,

$$0 = (64)^2 - 64\,s. \text{ (See p. 436.)}$$

Which gives $s = 64$ feet.

Ex. 4.—Find the greatest height ascended when the velocity of projection is 100, at an angle of 45°.

Ex. 5.—Find the velocity of projection at an angle of 45°, which is necessary in order to hit a mark on the ground half-a-mile off.

Ex. 6.—Find the angle of projection when the horizontal range is 100 feet, and the velocity of projection 100.

Ex. 7.—A body is projected horizontally from the top of a tower 100 feet high, and strikes the

ground 40 feet from its base; find the velocity of projection.

Here the time of flight (t) is the same as the time of falling down 100 feet.

$$\therefore\ 100 = 16\,t^2,\ \text{and}\ \therefore\ t = 2\tfrac{1}{2}.$$

Now 40 feet are described horizontally in this time; wherefore the velocity is

$$\frac{40}{2\frac{1}{2}}\ \text{or}\ 16.$$

Ex. 8.—If the body is projected horizontally with a velocity 20, how far will it fall from the base of the tower, the height being 64 feet?

Ex. 9.—If the body is projected at 45°, instead of horizontally, find the same.

MOTION DOWN A CURVE.

The velocity which a body acquires in moving down a smooth curve or groove, its weight being the only force causing the motion, is capable of being found by a very simple rule, which is investigated in the following Proposition.

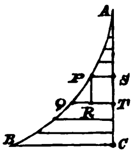
Proposition XVIII.

To find the velocity which a body acquires when it falls down along a smooth curve or groove.

Let $APQB$ be the curve or groove; and suppose that the body is let go at A, and allowed to run down this curve to B; it is required to find how fast it is moving at B.

Let PQ be any portion of the curve taken so small that it may be practically regarded as a straight line; draw PR and AC vertically; and BC, QT, and PS horizontally. The forces which act on the body (W) while it is moving from P to Q, are its weight W, and the reaction of the curve. This reaction is always at right angles to the curve, and therefore has no tendency to accelerate or retard the motion of W; also, if P denote the effect of W *down* PQ, we have, by Prop. XIII.,

Fig. 268.



$$P = W\frac{PR}{PQ},$$

and therefore,

$$\text{rate of acceleration } (f) = \frac{P}{W}g = \frac{PR}{PQ}g = \frac{ST}{PQ}g.$$

Let v denote the velocity of the body at P, and v' that at Q; then we have, by Prop. XII.,

$$v' = \sqrt{v^2 + 2f.PQ},$$

$$\text{or, } v' = \sqrt{v^2 + 2g.ST}, \left(\text{since } f = \frac{ST}{PQ}g\right).$$

Now suppose that the body, instead of moving down the curve, falls vertically down AC, and suppose also that v is its velocity at S; we then find, by Prop. XII., its velocity at T to be

$$\sqrt{v^2 + 2g.ST}.$$

Wherefore the increase of velocity that takes place when the body goes from P to Q in the actual case, is precisely the same as that which

takes place when the body goes from S to T in the supposed case. Hence, if we draw a great number of horizontal lines, as in the figure, thereby dividing the curve AB, and the vertical AC, in a series of corresponding elements, like PQ and ST, the change of velocity in the actual motion along the curve will be the same for each element, as that for the corresponding element in the supposed motion down AC. Wherefore, the final velocity will be the same in both motions, supposing the initial velocity the same. In other words, we have the following rule:—If a body be let fall from A down any smooth curve or groove AB, its velocity at B will be the same (in magnitude) as if it had been allowed to fall directly down the vertical height, *i.e.* from A to C.*

If then v denote the velocity at B, we have, by Prop. XII.,

$$v^2 = 2g.AC \quad \dots\dots\dots \quad (1).$$

Corollary.—In the same way it may be shown, that if the body, instead of being simply allowed to fall down AB, be projected along the curve with an initial velocity u, then,

$$v^2 = u^2 + 2g.AC \quad \dots\dots \quad (2).$$

The following is a good example of this, and Prop. XV.

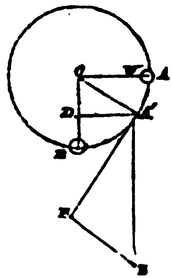
Proposition XIX.

A body (W), *fig.* 269, *is attached to one end of a horizontal string* AC, *the other end* C *being fixed. If the body be let go, to find its velocity when it gets to its lowest position* B, *and the tension of the string.*

* It may be shown that this proof is absolutely free from error, as in the proof that $s = \frac{1}{2} f t^2$.

It is clear that W, by the action of the string; is caused to describe a circular arc, $A\,B$, just as if it were simply allowed to move down the arc, as a smooth curve or groove. In going from A to B it descends a vertical distance equal to CB, which call r. Hence, if v denote the velocity at B, we have, by the previous proposition,

Fig. 269.



$$v^2 = 2gr \; . \; . \; . \; (1),$$

which gives v.

Also, the forces which act on W when at B, are the tension of the string (which call T) acting upwards, and the weight W downwards; *i.e.* $T - W$ acting towards the centre. Thus $T - W$ is the force which causes the body W to describe a circle whose radius is r, with a velocity v. Wherefore, by Prop. XV., we have,

$$T - W = \frac{W}{g}\,\frac{v^2}{r} \; . \; . \; . \; (2);$$

and, therefore, by (1),

$$T - W = \frac{W}{g}\,\frac{2gr}{r} = 2\,W;$$

$$\therefore\; T = 3\,W \; . \; . \; . \; . \; . \; (3).$$

The tension at B, therefore, is three times the weight.

Corollary 1.—If W be allowed to fall from a lower position A', instead of A, to find v and T.

Draw $A'D$ horizontally, and let $DB = h$, which we shall suppose is known. Then, as before,

$$v^2 = 2gh,$$

$$T - W = \frac{W}{g}\,\frac{v^2}{r} = \frac{W}{g}\,\frac{2gh}{r} = \frac{2Wh}{r},$$

$$\therefore\ T = \left(1 + \frac{2h}{r}\right) W;$$

Corollary 2.—The body being allowed to fall from A, to find v and T when it gets as far as A'.

We have,

$$v^2 = 2gCD = 2g(r - h),$$

which gives v.

To find T, we must resolve W (which represent by $A'E$) into two forces, one, $A'F$, at right angles to the string, and the other, FE, along the string.

Then $T - FE$ will be the whole force along the string.

Now by similar triangles, $CA'D$ and $A'FE$, we have,

$$FE : A'E :: CD : CA',$$

$$\text{or,}\quad FE : W :: r - h : r;$$

$$\therefore\ FE = W\,\frac{r-h}{r};$$

$$T - W\,\frac{r-h}{r} = T - FE = \frac{W}{g}\,\frac{v^2}{r} = \frac{W}{g}\,\frac{2g(r-h)}{r},$$

$$\therefore\ T = 3W\,\frac{r-h}{r}.$$

CHAPTER V.

SIMPLE IMPACT.

IMPACT is said to take place when bodies strike together; we shall here consider only the simplest case in which this may occur.

PROPOSITION XX.

If a body A overtake and strike another body B, to find the effect of the blow or impact: both bodies are supposed to be moving along the same line.

Let u denote the velocity of A, and v that of B, both taking place in the same direction, but that of A being the greater. When A overtakes, and begins to press upon B, it is clear that the velocity

Fig. 270.

of B will be increased by that pressure. At the same time, B will react upon A with an equal and opposite pressure, which will, of course, diminish the velocity of A. Now, as long as A is moving faster than B, this increase of B's velocity and decrease of A's velocity will continue. Wherefore, at last, the velocities of A and B will be made equal. Let v' be this equalized velocity;

then A has lost an amount of velocity equal to $u - v'$; and therefore, by Prop. VI., the *dynamical effect* produced on A by the pressure of B is

$$\frac{A}{g}(u - v'),$$

supposing A to denote the weight of A.

In like manner, the alteration of velocity of B is $v' - v$ (observe that v' is greater than v); and therefore the dynamical effect on B is

$$\frac{B}{g}(v' - v),$$

B denoting the weight of B.

Now, the pressures which produce these dynamical effects, namely, the pressure of B on A, and that of A on B, are, by the law of action and reaction, equal during the whole time while the alteration of velocities is going on. The dynamical effects produced must therefore be equal. Consequently we have,

$$\frac{B}{g}(v' - v) = \frac{A}{g}(u - v');$$

and $\therefore$ $(B + A)\, v' = A\, u + B\, v.$

$$\text{Or, } v' = \frac{A\, u + B\, v}{A + B} \quad \ldots\ldots \quad (1).$$

This gives the equalized velocity v'.

If the bodies be perfectly devoid of spring, this equation completely solves the problem; for when once the velocities are equalized, it is clear that there can be no mutual pressure exercised between the two bodies. But all bodies are compressible,

and when compressed have a greater or less spring, or tendency to recover their shapes. It is clear that A and B will be in a state of compression when the velocities are equalized; for they have been mutually pressing on each other, and that with no small amount of force. Wherefore the spring, or tendency to recover shape, will come into play after the equalization of the velocities, and cause a new exertion of mutual pressure between A and B; which will further increase the velocity of B and diminish that of A, until B gets clear of and leaves A behind. Then the impact will be complete.

Let u'' be the velocity of A, and v'' that of B at this time, that is, when B gets clear of A. Then it is found by experiment, that the alteration of velocity, $v'-u''$, produced in A by the spring, is always a certain fraction of the alteration $u-v'$ previously produced, while the compression was going on. That fraction is always less than 1, and its amount depends upon the *elasticity*, as it is called, of the two bodies. Let the letter λ be employed to denote this fraction; then we have,

$$v'-u''=\lambda\ (u-v').$$

$$\therefore\ u''=(1+\lambda)v'-\lambda u;$$

or, by (1),

$$u''=(1+\lambda)\frac{Au+Bv}{A+B}-\lambda u\ .\ .\ .\ .\ .\ (2).$$

It is found also, similarly, that the same law applies to the alteration of velocity in B; that is,

$$v''-v'=\lambda(v'-v),$$

$$\therefore\ v'' = (1+\lambda)\, v' - \lambda v,$$

or, by (1),

$$v'' = (1+\lambda)\, \frac{Au + Bv}{A+B} - \lambda v \ .\ .\ .\ .\ .\ (3).$$

(2) and (3) give the velocities u'' and v'', and thus the final effect of the impact is known.

Corollary.—If B be moving in the opposite direction to A, we must in all the above formulæ *change the sign* of v, as is manifest. Thus, instead of (2) and (3), we shall, in this case, have,

$$u'' = (1+\lambda)\, \frac{Au - Bv}{A+B} - \lambda u \ .\ .\ .\ .\ .\ (4).$$

$$v'' = (1+\lambda)\, \frac{Au - Bv}{A+B} + \lambda v \ .\ .\ .\ .\ .\ (5).$$

Observation.—When $\lambda = 0$, the bodies are said to be *inelastic;* when $\lambda = 1$, they are said to be *perfectly elastic.* Neither of these cases occur in nature.

Ex. 1.—A, moving with velocity 10, strikes B at rest; $A = B$, and $\lambda = \frac{1}{2}$; find u'' and v''.

Here $u = 10$, $v = 0$; wherefore, by (2) and (3),

$$u'' = \frac{3}{2}\,\frac{A.10}{2A} - \frac{1}{2}\,10 = 2\tfrac{1}{2};$$

$$v'' = \frac{3}{2}\,\frac{A.10}{2A} = 7\tfrac{1}{2}.$$

Wherefore, A loses $7\frac{1}{2}$ of its velocity, and B gains as much.

Ex. 2.—Same case, only $B=2A$; find u'' and v''. We have

$$u''=\frac{3}{2}\frac{A.10}{3A}-\frac{1}{2}10=0;$$

$$v''=\frac{3}{2}\frac{A.10}{3A}=5.$$

Here A is completely stopped, and B goes on with the velocity 5.

Ex. 3.—Same case, only $B=3A$; find u'' and v''. Here

$$u''=\frac{3}{2}\frac{A.10}{4A}-\frac{1}{2}10=-1\tfrac{1}{4};$$

$$v''=\frac{3}{2}\frac{A.10}{4A}=3\tfrac{3}{4}.$$

Here the negative sign of u'' shows that A is not only stopped, but driven back by the blow.

Ex. 4.—Show that, if $\lambda=1$, and B is at rest before impact, A is always at rest after impact.

Ex. 5.—If $A=2B$, and A and B are moving in opposite directions before impact, find the final velocities.

Ex. 6.—Show that always

$$Au+Bv=Au''+Bv''.$$

Proposition XXI.

If B be a fixed body, to find the velocity of A after impact. Fig. 270.

In this case we may find the result by supposing B, in the former proposition, to be exceedingly large compared with A. Thus, if I throw a heavy body on the ground, I do not move the earth, because it is so large; not because it is actually fixed.

Now we have from (2), putting $v=0$,

$$u''=(1+\lambda)\frac{A}{A+B}u-\lambda u.$$

But, because B is exceedingly large compared with A, the denominator of the fraction $\dfrac{A}{A+B}$ is extremely large compared with its numerator. The fraction is therefore practically zero. Wherefore we find,

$$u''=-\lambda u;$$

that is, the velocity of A is reversed by the blow, and diminished in the proportion of λ to 1.

Ex. 1.—A marble is dropped on the floor from a height h, and rebounds to a height $\frac{9}{16}h$; find λ.

The ball, since it falls a height h, strikes the ground with a velocity $\sqrt{2gh}$, and, since it rises

to a height $\frac{9}{16}h$, it leaves the ground with a velo-$\sqrt{2g\frac{9}{16}h}$, or $\frac{3}{4}\sqrt{2gh}$. Wherefore,

$$u = \sqrt{2gh},\quad u'' = -\frac{3}{4}\sqrt{2gh}.$$

$$\therefore \lambda = \frac{3}{4}.$$

Ex. 2.—If $\lambda = \frac{1}{2}$, and the ball is dropped from a height h, find the height to which it rises after 2 rebounds.

Here let h' and h'' be the heights to which the ball rises after the first and second rebound; then

$$\sqrt{2gh'} = \lambda\sqrt{2gh},\ \text{and}\ \sqrt{2gh''} = \lambda\sqrt{2gh'}.$$

$$\therefore h' = \lambda^2 h = \frac{1}{4}h,\ \text{and}\ h'' = \frac{1}{4}h'.$$

$$\therefore h'' = \frac{1}{16}h.$$

OBLIQUE IMPACT ON A PLANE.

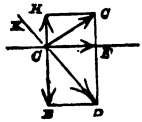
Proposition XXII.

If A be thrown against the ground obliquely, with a velocity u, to find how it moves after the blow.

Let CF be the plane, and suppose that A is thrown against it in the direction KC. Produce

KC to D, taking CD to represent the velocity u. Draw HCE at right angles to CF; and complete the rectangle $CEDF$. Take $CH = \lambda CE$; complete the rectangle $CHGF$, and draw CG. Then CG represents in magnitude and direction the velocity of A after the impact.

Fig. 271.



For the velocity CD is equivalent to the two velocities CF and CE; the former clearly is not affected nor altered by the impact; the latter, CE, is reversed and changed into λCE; in other words, it is changed into CH (for $CH = \lambda CE$). Wherefore, after impact, A has the two velocities CH and CF; and consequently the actual velocity of A is represented by CG.

Corollary.—The angle DCE is called the *angle of incidence*, and GCH the *angle of reflection*. Now, $\tan. DCE = \frac{DE}{CE}$, and $\tan. GCH = \frac{GH}{CH} = \frac{DE}{\lambda CE}$;

$$\therefore \tan. GCH = \frac{1}{\lambda} \tan. DCE.$$

This completes the explanation and illustration of the fundamental principles of Dynamics.

END OF VOL. I.

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